Fermion Doubling in QCA

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Quantum Computation Structures(QuaCS)

Outline

- Brief introduction to fermions
- Quantum Cellular Automaton (QCA) for quantum electrodynamics (QED)
- Fermion doubling problem
- A solution for the doubling problem
- Relationship with topology

The Paradigm of physics back then and today

1750s - 1910s : Finding the right equations of motions of a given system that explains the physical phenomenon that is being observed.

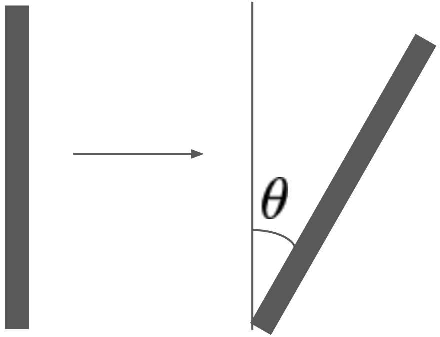
Mathematical framework: Differential equations, optimization theory

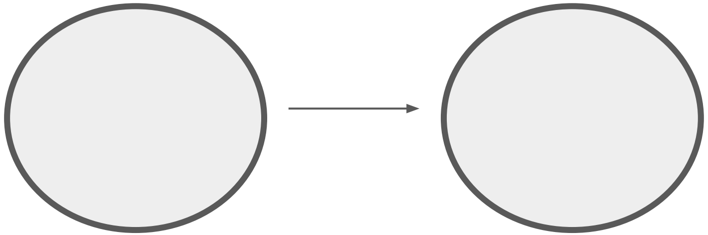
1910s - Today : Finding and understanding the right symmetries of a given system that would allow us to design experiments and understand observations

Mathematical framework: Group theory, Geometry

Symmetries and representations

They preserve a quantity, for example: SO(2) preserves the length of a given object





$$\rho_r(r,\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\rho_0(r,\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Fermions in (1+1)-D

Definition: Fermionic field in (1+1)D is defined by the following data:

1) $\psi : \mathbb{R}^2 \to \mathbb{C}^2$ *is a smooth map from reals to complex*

$$which\ has\ \ \psi(x,t)\psi(x',t') = -\psi(x',t')\psi(x,t)\ \ for\ (x,t) \in \mathbb{R}^2$$

2) *Solution of the Dirac equation* $(i\gamma^0\partial_t + i\gamma^1\partial_x - m\mathbb{1})\psi(x,t) = 0$

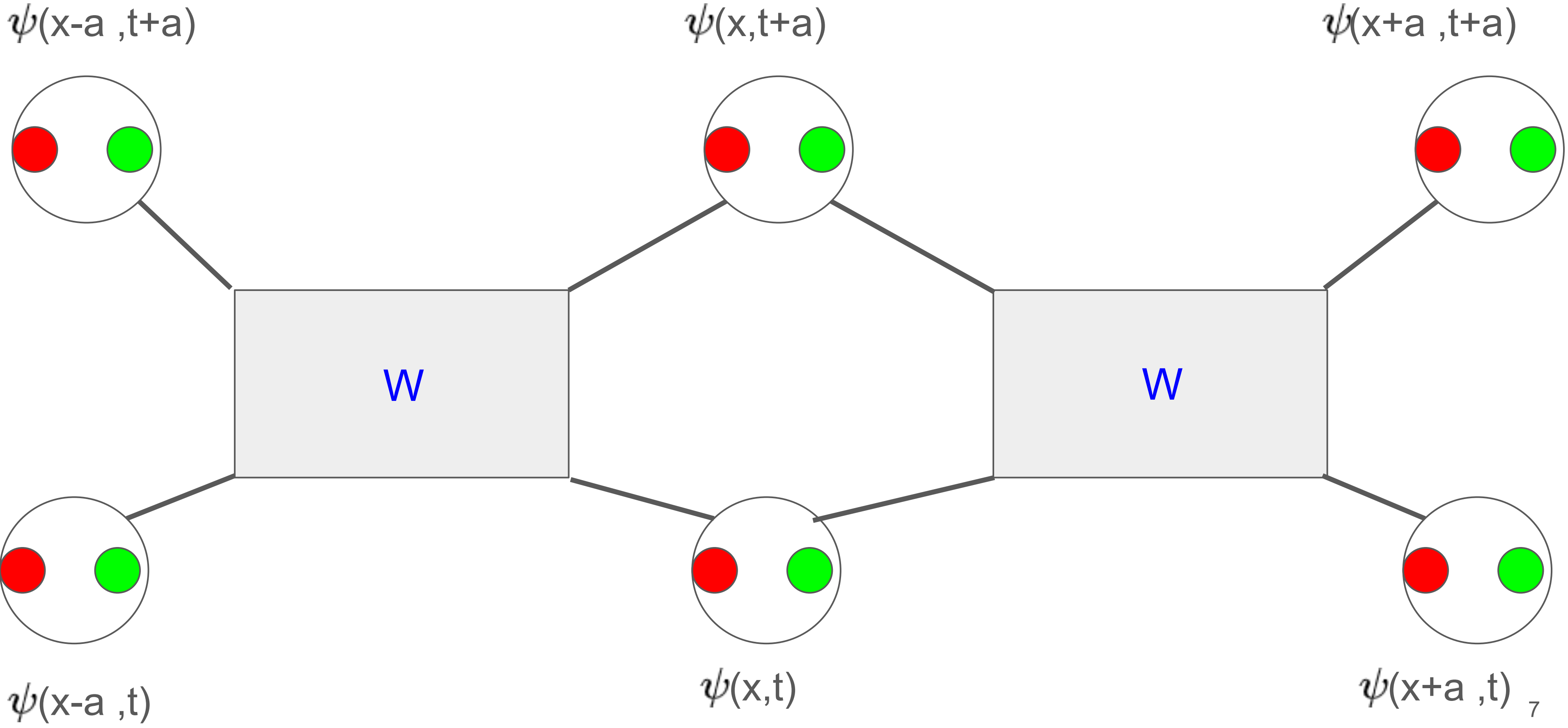
$$where,\ \gamma^0\gamma^1 = -\gamma^1\gamma^0,\ and\ \gamma^0\gamma^0 = -\gamma^1\gamma^1 = \mathbb{1}$$

SU(2) symmetry in Dirac Equation

$$(i\gamma^0\partial_t + i\gamma^1\partial_x - m\mathbb{1})(\mathcal{M}\psi(x,t)) = (i\gamma^0\partial_t + i\gamma^1\partial_x - m\mathbb{1})\psi(x,t)$$

$$\mathcal{M} \in SU(2),\ 2\times 2\ unitary\ matrix\ with\ \ \det\mathcal{M} = \pm 1$$

Quantum Cellular Automaton for QED in (1+1)D



$$\hat{U}\psi(x,t) = \begin{pmatrix} \cos ma\,\hat{S} & -i\sin ma \\ -i\sin ma & \cos ma\,\hat{S}^\dagger \end{pmatrix} \begin{pmatrix} \psi^{r}(x,t) \\ \psi^{g}(x,t) \end{pmatrix}$$

$$= \begin{pmatrix} \psi^{r}(x,t+a) \\ \psi^{g}(x,t+a) \end{pmatrix}$$

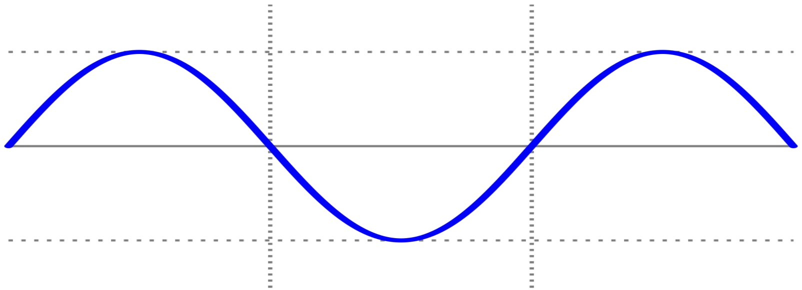
In the limit $a \to 0$:

$$(i\gamma^0\partial_t + i\gamma^1\partial_x - m\mathbb{1})\psi(x,t) = 0$$

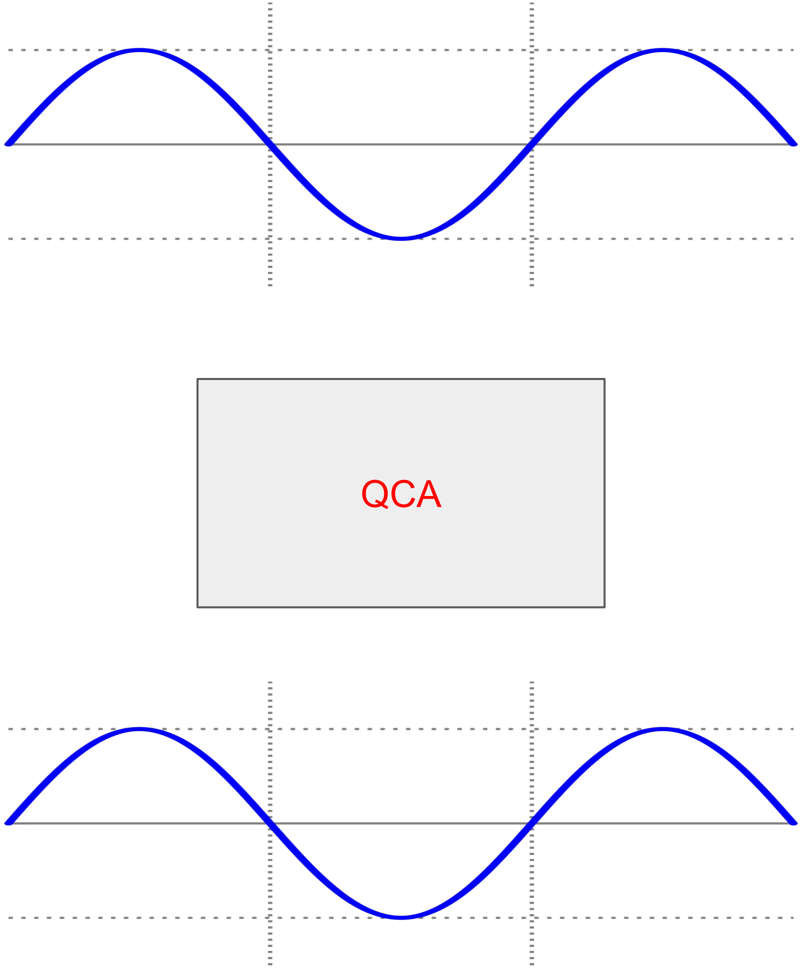
$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

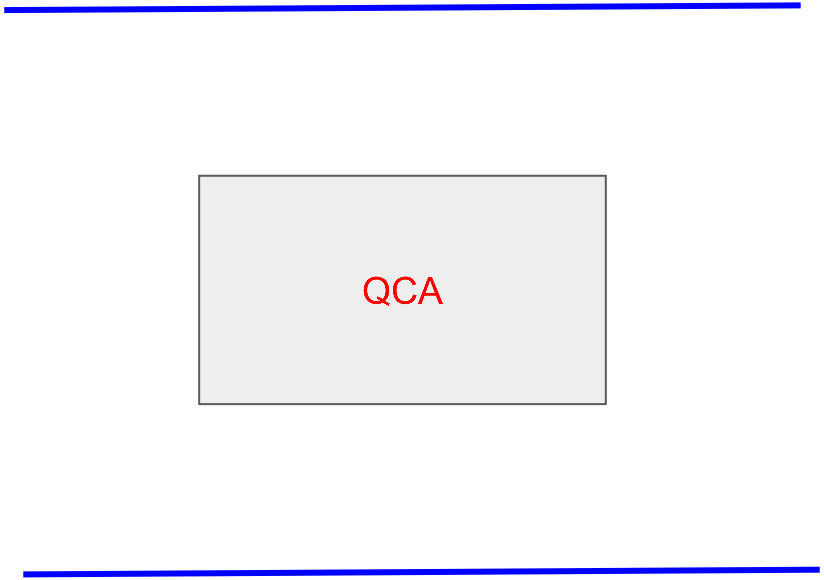
Fermion Doubling

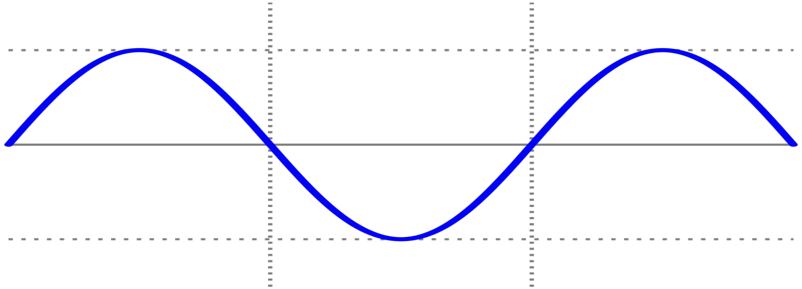
$$\psi^H(x,t) = \tfrac{1}{a}\begin{pmatrix}(-1)^{(x+t)/a} \\ (-1)^{(x-t)/a}\end{pmatrix} =$$



$$\psi^L(x,t) = \tfrac{1}{a}\begin{pmatrix}1 \\ 1\end{pmatrix} =$$

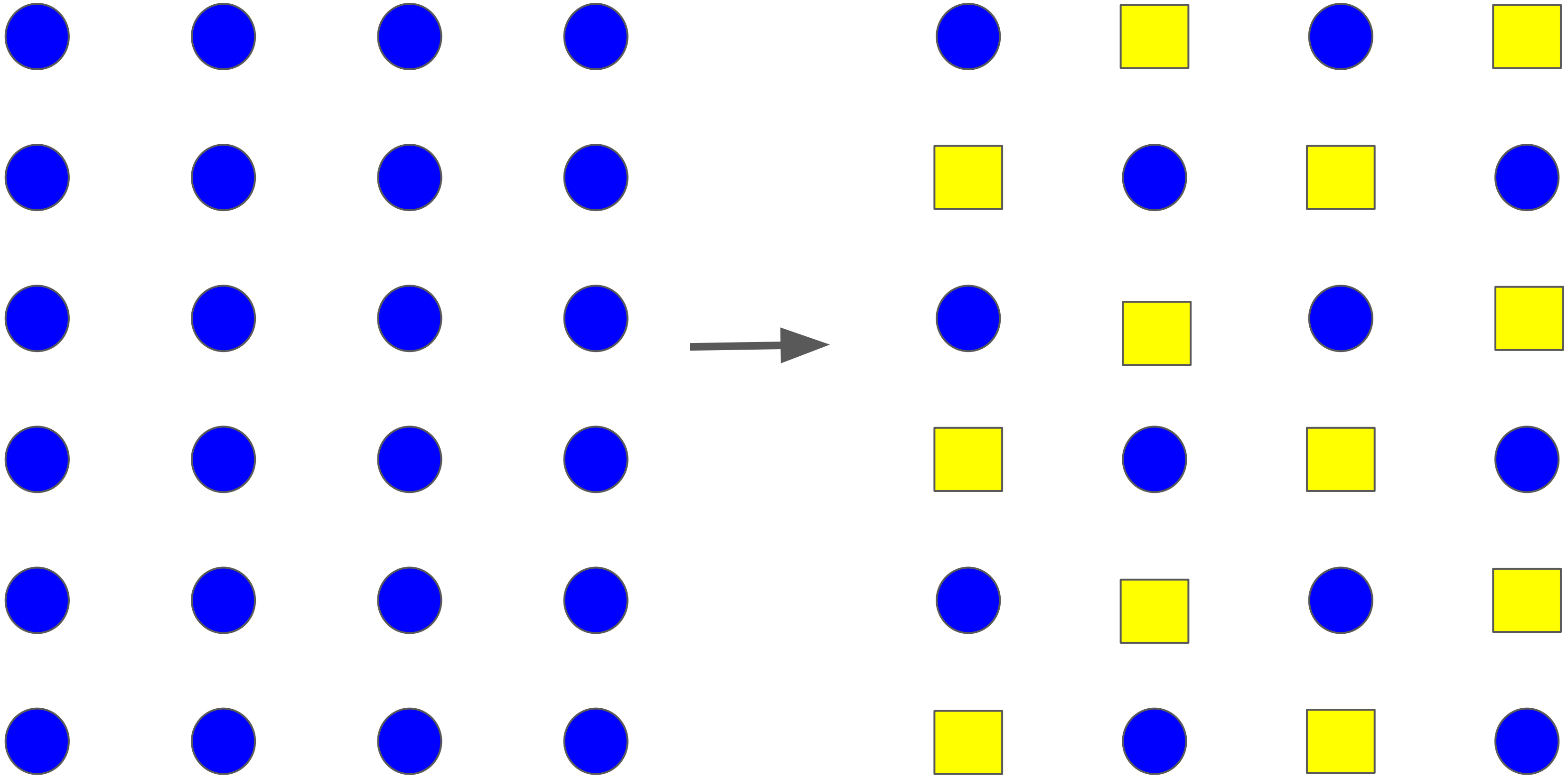

QCA


QCA

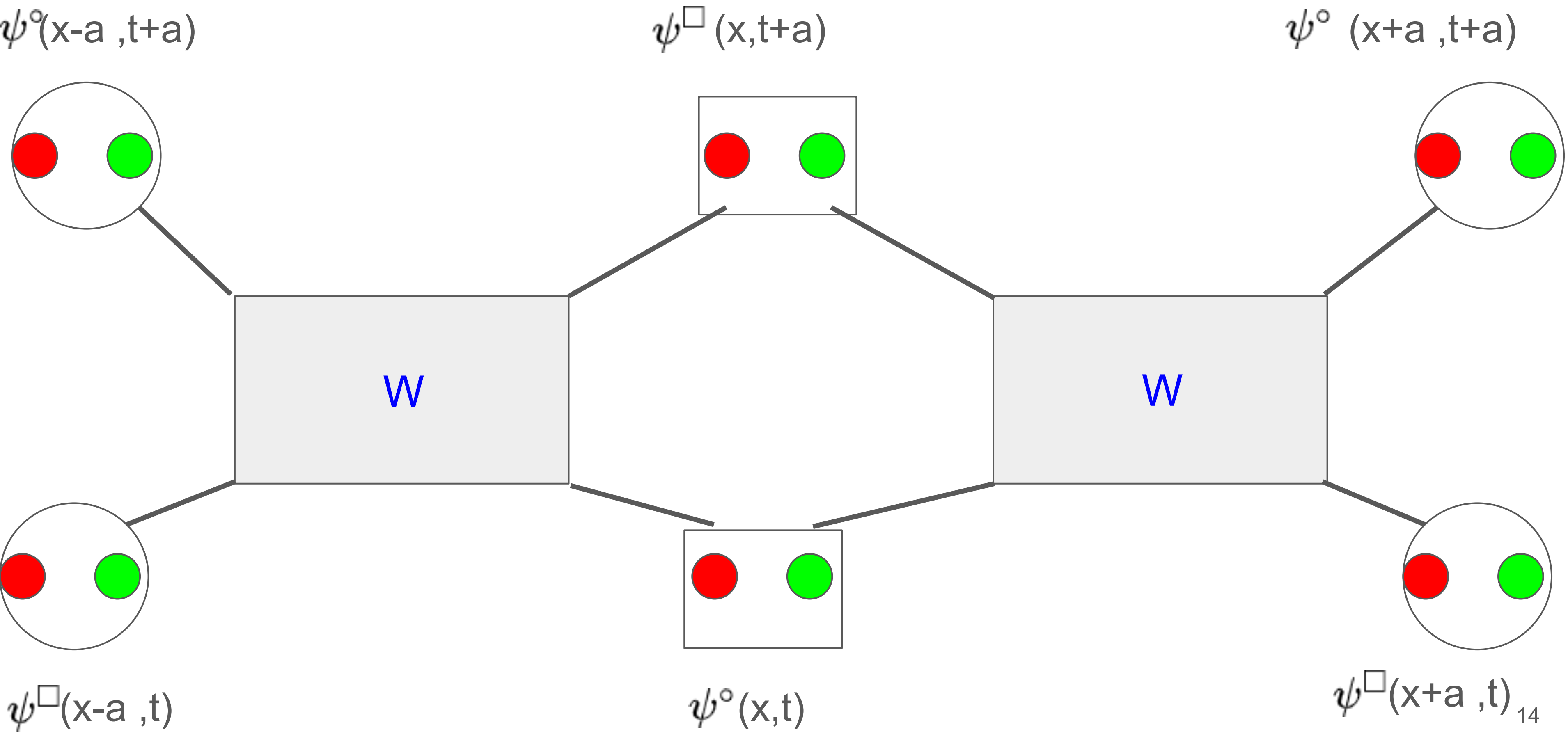


is symmetric to by QCA

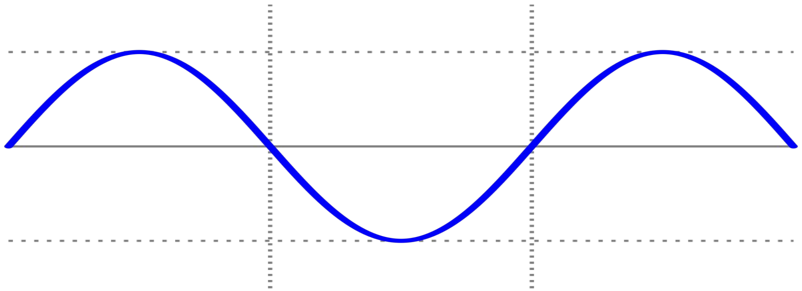
This symmetry is non-physical, which allow ultra fast particle to behave like very slow particles. We do not have such a symmetry in nature.

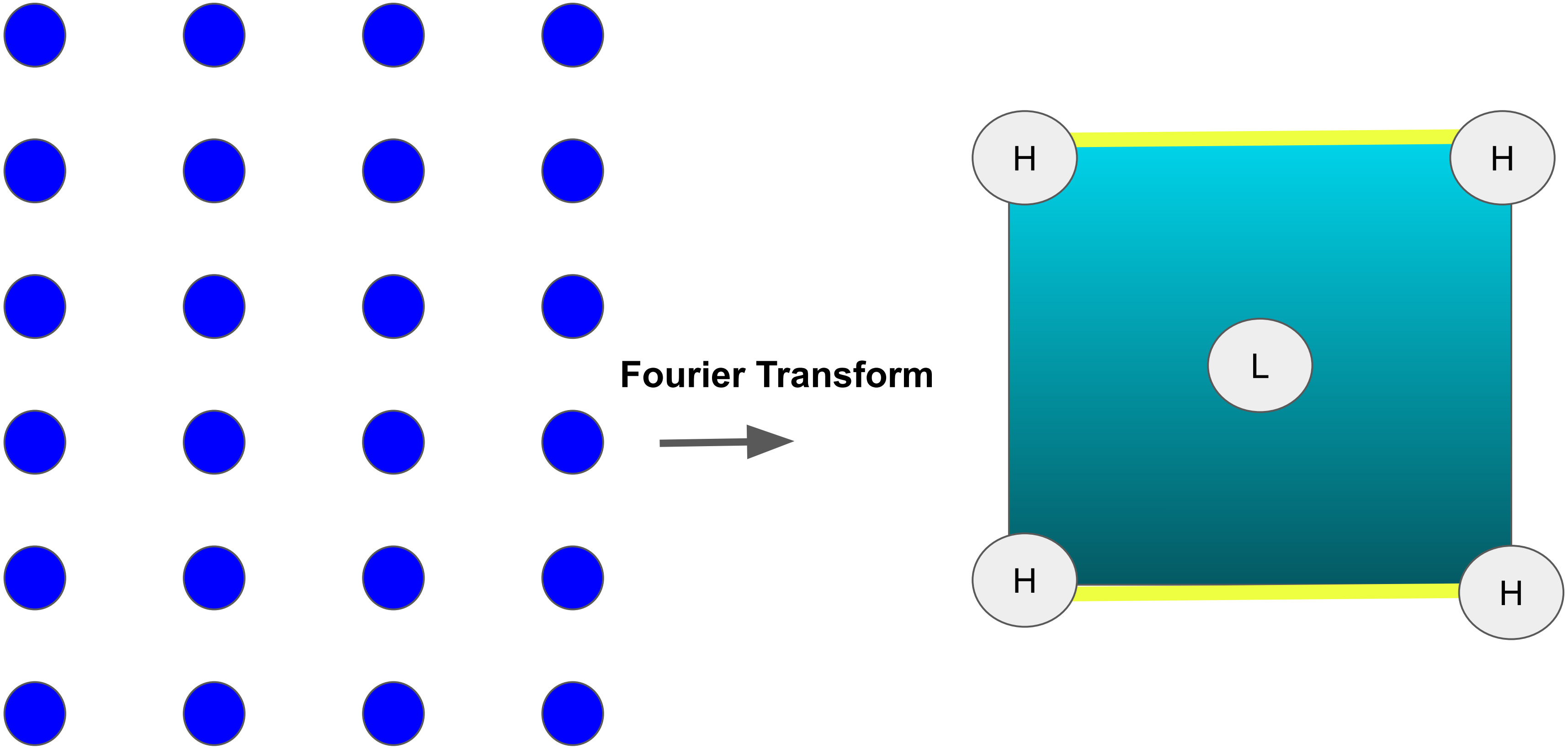


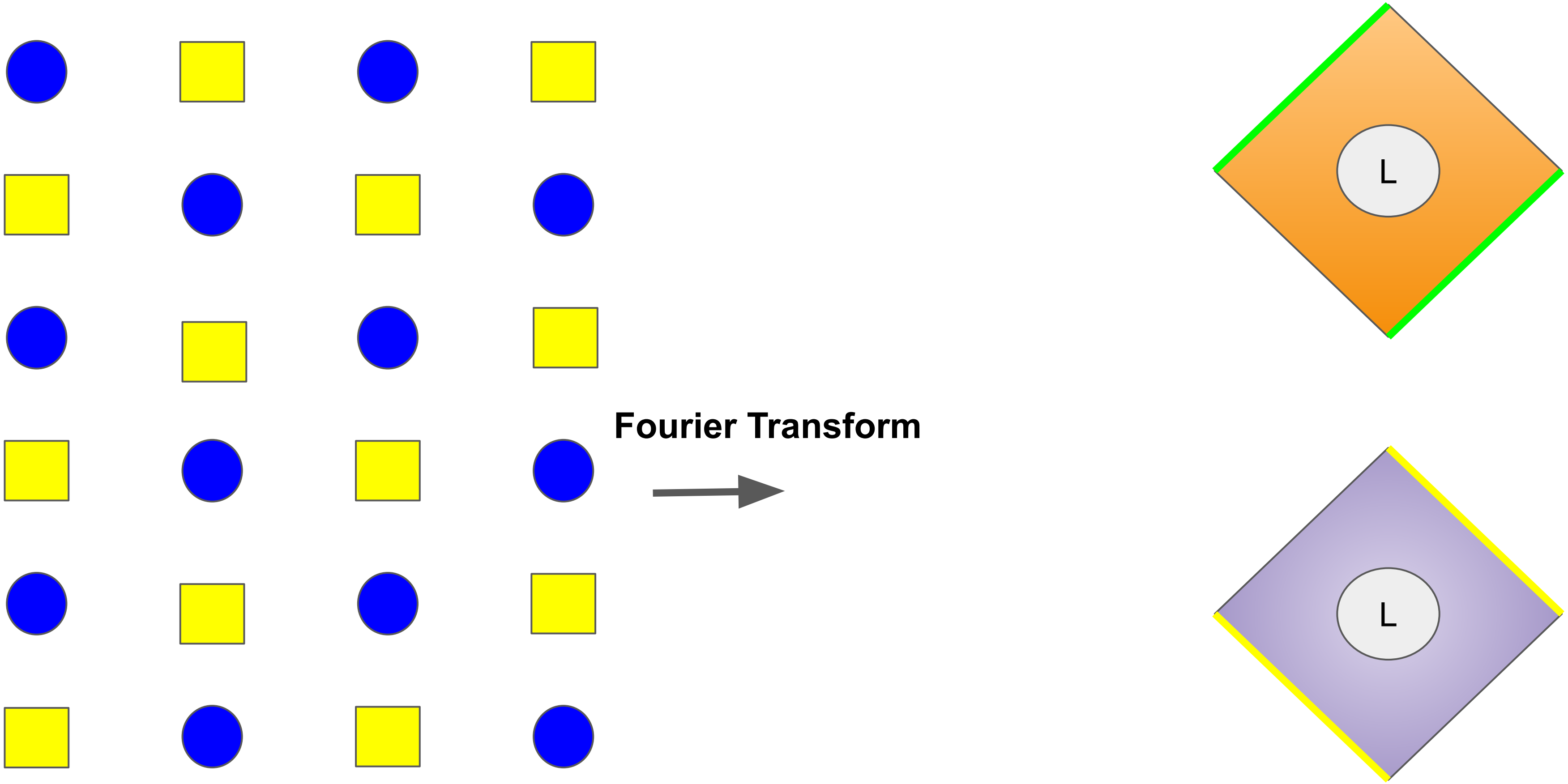
Solution of fermion doubling



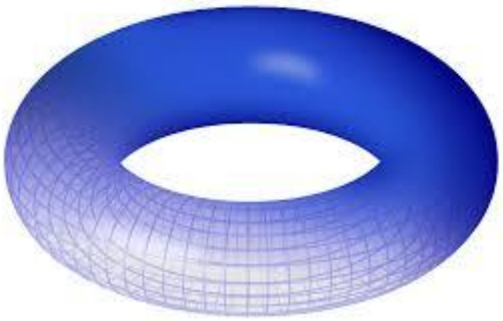
New lattice geometry forbids the existence of



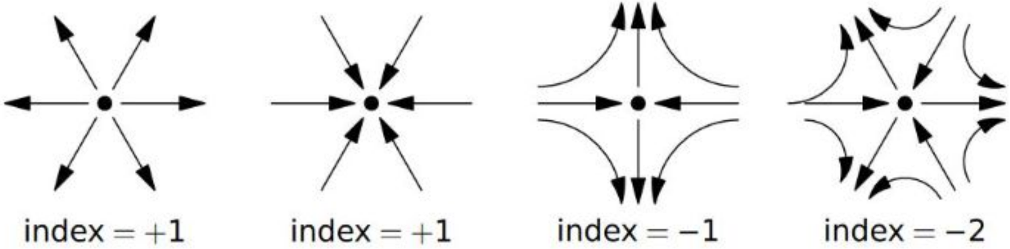

Fourier Transform
H
H
L
H
H


Fourier Transform
L
L

$\sim$

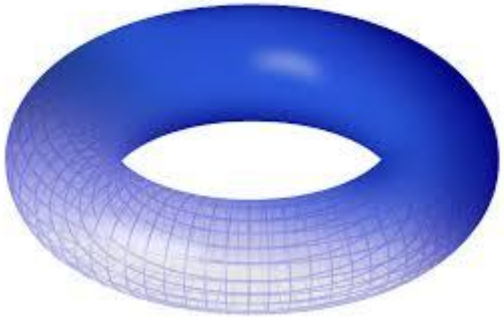


$$\chi(X) \coloneqq \chi[TX] = \sum_{\substack{\text{isolated zero} \\ x_i}} \operatorname{ind}_{x_i}(v)\,.$$



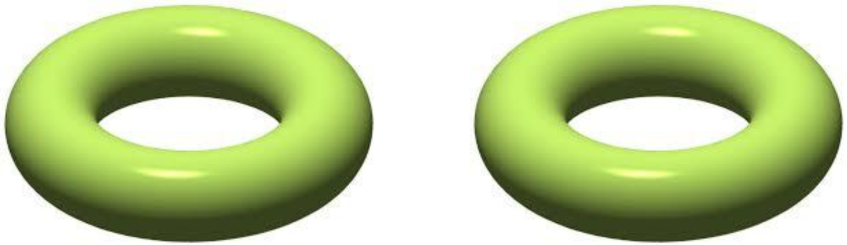
We realize that fermion doubling is due to the topology of the Fourier space of lattice. When we define a vector field on a torus with an index i, another vector field with an index -i is created due to the topology of the torus.

This is not the case in continuum because the Fourier space of space-time manifold is not compact, hence Poincare-Hopf theorem does not put restrictions on the vector fields that are defined on the Fourier space of space-time manifold



QCA

$$\mathcal{H}^L \oplus \mathcal{H}^H$$



QCA

$$\mathcal{H}^\circ \oplus \mathcal{H}^\square$$