The (big) numbers game

Laboratoire Méthodes Formelles

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November 5th 2024

Why do we care?

Why do we care?

COMPETITION!

Why do we care?

The duel between Agustín Rayo and Adam Elga went like this:

- 1
- 111111111111111111111111
- $11!!!!!!!!!!!!!!!!!!!!!!$
- $\mathrm{BB}\left(10^{100}\right)$
- Busy Beaver hierarchy $\Rightarrow \mathrm{BB}_{\theta}\left(10^{100}\right)$
- Rayo's number

Outline

- Knuth's arrows
- Fast-growing hierarchy
- Busy beavers
- Rayo's number

Outline

- Knuth's arrows → Primitive recursive functions
- Fast-growing hierarchy → General recursion
- Busy beavers → Non-computable functions
- Rayo's number → Second-order logic

Knuth’s up-arrow notation

Iterate on operations

- addition: $100 + 100$
- multiplication: 100×100
- exponentiation: 100^{100}
- tower of exponentiations: $100^{\overbrace{\left(100^{100^{\cdot^{\cdot^{\cdot^{100}}}}}\right)}_{100 \text{ times}}}$

Knuth's up-arrow notation

$$a \uparrow b \triangleq a^b$$

$$a \uparrow\uparrow b \triangleq \underbrace{a \uparrow (a \uparrow (a \uparrow \ldots))}_{b \text{ copies of } a}$$

$$\ldots$$

$$a \uparrow^n b \triangleq \underbrace{a \uparrow^{n-1} (a \uparrow^{n-1} (a \uparrow^{n-1} \ldots))}_{b \text{ copies of } a}$$

Examples

$$\text{gogol} = 10^{100} = 10 \uparrow 100$$

$$\text{gogolplex} = 10^{10^{100}} < 10 \uparrow\uparrow 4$$

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Graham’s number

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$$g_1 = 3 \uparrow\uparrow\uparrow\uparrow 3$$

Graham's number

$$g_2 = 3 \underbrace{\uparrow\uparrow \dots \uparrow}\; 3$$

$$g_1 = 3 \uparrow\uparrow\uparrow\uparrow 3$$

Graham's number

$$g_3 = \quad 3 \underbrace{\uparrow\uparrow \dots\dots\dots \uparrow}\ 3$$

$$g_2 = \quad 3 \underbrace{\uparrow\uparrow \dots \uparrow}\ 3$$

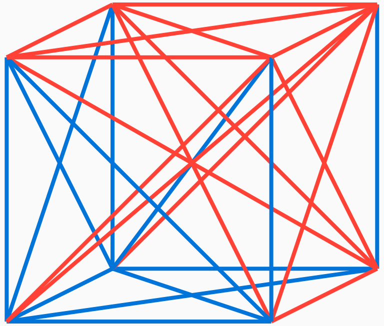
$$g_1 = \quad 3 \uparrow\uparrow\uparrow\uparrow 3$$

Graham's number

$$
\begin{aligned}
G = g_{64} &= 3 \underbrace{\uparrow\uparrow \cdots\cdots\cdots\cdots \uparrow}_{\displaystyle 3 \underbrace{\uparrow\uparrow \cdots\cdots\cdots \uparrow}_{\displaystyle \vdots} 3} 3 \\
g_3 &= 3 \underbrace{\uparrow\uparrow \cdots\cdots \uparrow} 3 \\
g_2 &= 3 \underbrace{\uparrow\uparrow \cdots \uparrow} 3 \\
g_1 &= 3 \uparrow\uparrow\uparrow\uparrow 3
\end{aligned}
$$

What is the use of g_{64}?

Take a coloring of an n-dimensional cube, and try to extract a single-colored subgraph of size 4:



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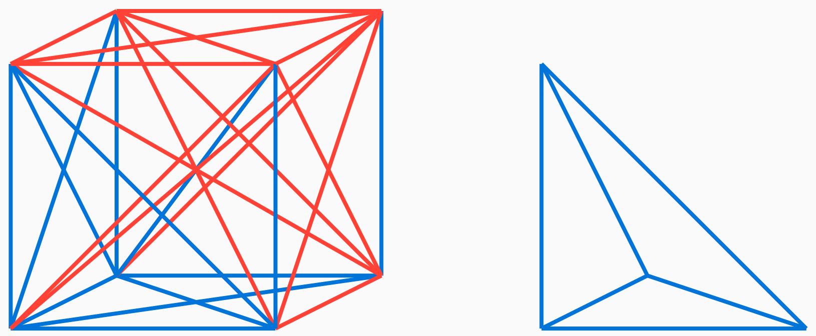

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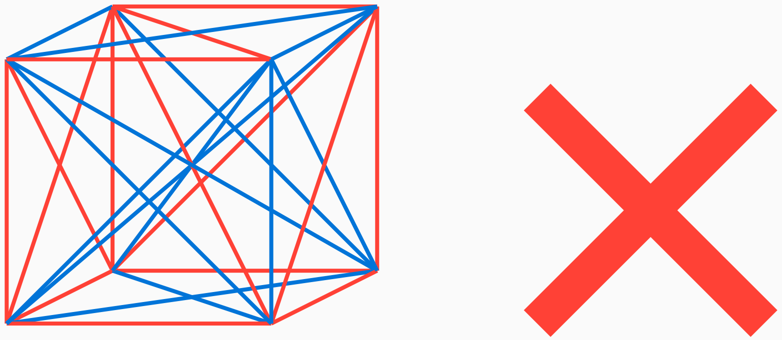

What is the use of g_{64}?

Find the smallest n, such that:

If you take the n-dimensional cube, and consider the complete graph of its vertices, any 2-coloring of the edges of this graph contains a complete subgraph of size 4 with a single color.

$$< n <$$

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If you take the n-dimensional cube, and consider the complete graph of its vertices, any 2-coloring of the edges of this graph contains a complete subgraph of size 4 with a single color.

$$13 < n < 2 \uparrow\uparrow (2 \uparrow\uparrow 5138)$$

Fast-growing functions hierarchy

The fast-growing functions hierarchy

A family of functions $f_\alpha(n)$, where α is an *ordinal*.

Reminder on ordinals

- $0, 1, 2, 3, \ldots$
- $\omega = \{0, 1, 2, \ldots\}$ $\rightarrow$ 
- $\omega + 1 = \omega \cup \{\omega\}$ $\rightarrow$ 

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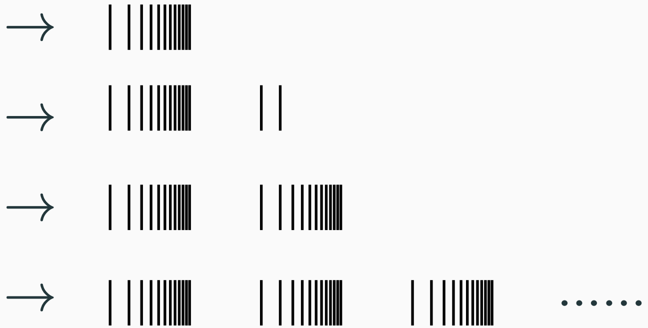
- $2\omega = \{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots\}$
- $\omega^2 = \{\dots, 2\omega, 3\omega, \dots\}$
-
-



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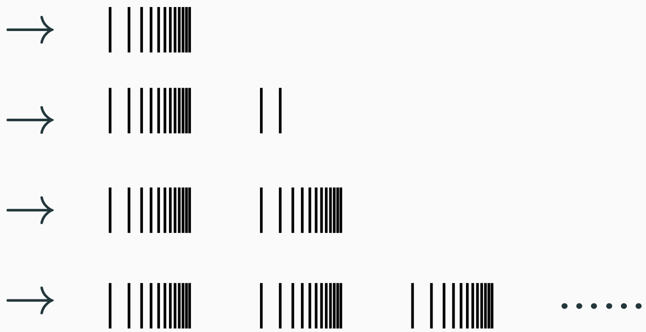
- $\omega^\omega = \{\ldots, \omega^2, \omega^3, \ldots\}$
-



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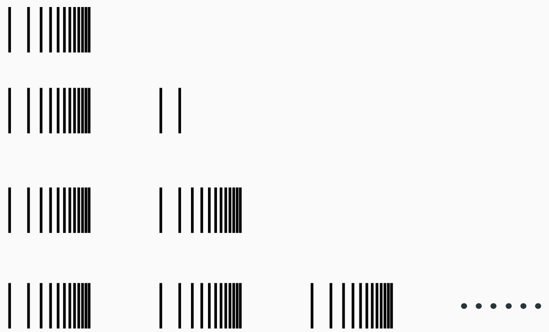
- $2\omega = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots\}$ $\rightarrow$
- $\omega^2 = \{\ldots, 2\omega, 3\omega, \ldots\}$ $\rightarrow$
- $\omega^\omega = \{\ldots, \omega^2, \omega^3, \ldots\}$
- $\omega^{\omega^\omega + 3} + \omega^2 + 5$



Some f_α functions

- $f_0(n) = n + 1$
- $f_1(n) =$
- $f_2(n) =$
- $f_\omega(n) =$
- $f_{\omega+1}(n) =$

Some f_α functions

- $f_0(n) = n + 1$
- $f_1(n) = (f_0)^n(n) = f_0(f_0(...f_0(n))) = 2n$
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- $f_{\omega+1}(n) = f_\omega(f_\omega(...f_\omega(n)))$

Some f_α functions

- $$\begin{aligned} f_{2\omega}(n) &= f_{\omega+\omega}(n) \\ &= f_{\omega+n}(n) \end{aligned}$$
- $$\begin{aligned} f_{\omega^2}(n) &= f_{n\times\omega}(n) \\ &= f_{(n-1)\times\omega+n}(n) \end{aligned}$$
- $f_{\omega^\omega}(n) = f_{\omega^n}(n)$
- etc.

Definition of f_α

Let α be an ordinal (smaller than ε_0).

- $f_0(n) \triangleq n + 1$
- $f_{\alpha(n)}$:
  - If $\alpha = \alpha' + 1$, $f_{\alpha(n)} \triangleq (f_{\alpha'})^n(n)$
  - Else, $f_{\alpha(n)} \triangleq f_{\alpha[n]}(n)$, where →

- $\left(\omega^{\beta_0} + \omega^{\beta_1} + \ldots + \omega^{\beta_k}\right)[n] \triangleq$ $\omega^{\beta_0} + \omega^{\beta_1} + \ldots + \left(\omega^{\beta_k}[n]\right)$
- If $\beta = \beta' + 1$, $\omega^{\beta}[n] \triangleq n \times \omega^{\beta'}$
- Else, $\omega^{\beta}[n] \triangleq \omega^{\beta[n]}$

Examples

$$f_{\omega^2}(2)$$

Examples

$$
\begin{aligned}
f_{\omega^2}(2) &= f_{2\omega}(2) \\
&= f_{\omega+2}(2)
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f_{\omega^2}(2) &= f_{2\omega}(2) \\
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&= f_{\omega+1}(f_{\omega}(f_{\omega}(2)))
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&= \ldots
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$$

Examples

$$f_\omega(n) = f_n(n)$$

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$f_\omega(n) = f_n(n) \to \quad \approx$ Knuth's arrows !

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$$f_{\omega+1}(n) = f_\omega(f_\omega(...f_\omega(n)))$$

Examples

$$f_\omega(n) = f_n(n) \rightarrow \quad \approx \text{Knuth's arrows !}$$

$$f_{\omega+1}(n) = f_\omega(f_\omega(...f_\omega(n))) \rightarrow \text{Graham's number !}$$

$$f_{\varepsilon_0}(n) = f_{\underbrace{\omega^{\omega^{\cdots^{\omega}}}}_{n \text{ times}}}(n)$$

Busy beavers and oracles

Busy beavers

The second-to-last of Rayo's propositions was $\mathrm{BB}\left(10^{100}\right)$: what is that?

Busy beavers

Take all turing machines that:

- Have the alphabet $\{0, 1\}$
- Have n or less states
- Run on a originally all 0 tape
- Terminate

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→ This already beats our previous solutions!

Oracles

Now add an *oracle* : a black box, that says whether a normal Turing machine terminates or not.

With those "second-order" Turing machines, we can define $\mathrm{BB}_2(n)$.

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Then $\mathrm{BB}_3(n)$, $\mathrm{BB}_4(n)$, $\mathrm{BB}_5(n)$, ...

Then $\mathrm{BB}_\omega(n)$...

Same as before, we need to stop eventually: Adam Elga stopped at $\mathrm{BB}_\theta\left(10^{100}\right)$.

Winner?

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So, can we even go beyond that in any meaningful way?

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So, can we even go beyond that in any meaningful way? *yes!*

Winner

We generalized over the notion of computation : it’s time to generalize over logic itself!

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Rayo's number is the biggest number that can be defined with a first-order formula of size 10^{100}.

The formula

If we have a function $[\cdot] : \text{formula} \to \mathbb{N}$, define the second-order formula $\mathrm{Sat}([\varphi], s)$ to mean

$$\begin{aligned}
&\text{for all } R, \{\forall \psi, \forall t \text{ a variable assignement},\\
&\qquad R([\psi], t) \leftrightarrow \big(\big(\psi = \text{"}x_i \in x_j\text{"} \wedge t(x_i) \in t\big(x_j\big)\big) \vee \\
&\qquad\qquad \big(\psi = \text{"}x_i = x_j\text{"} \wedge t(x_i) = t\big(x_j\big)\big) \vee \\
&\qquad\qquad (\psi = \text{"}\neg\theta\text{"} \wedge \neg R([\theta], t)) \vee \\
&\qquad\qquad (\psi = \text{"}\theta_1 \wedge \theta_2\text{"} \wedge R([\theta_1], t) \wedge R([\theta_2], t)) \vee \\
&\qquad\qquad (\psi = \text{"}\exists x_i \theta\text{"} \wedge \exists X, R([\theta], t[x_i \leftarrow X]))\big)\\
&\quad\} \to R([\varphi], s)
\end{aligned}$$

The formula

Then Rayo's number is

$$
\begin{aligned}
R := \max\Big(\Big\{m \in \mathbb{N} \mid\ & \exists \varphi \in \mathrm{arity}(1), \\
& |\varphi| \leq 10^{100} \\
& \exists s, \mathrm{Sat}([\varphi], s) \wedge s(x_1) = m \\
& \forall t, \mathrm{Sat}([\varphi], t) \rightarrow t(x_1) = m \\
\Big\}\Big) &
\end{aligned}
$$

Thank you!