LMF PhD
Seminar 2024-2025

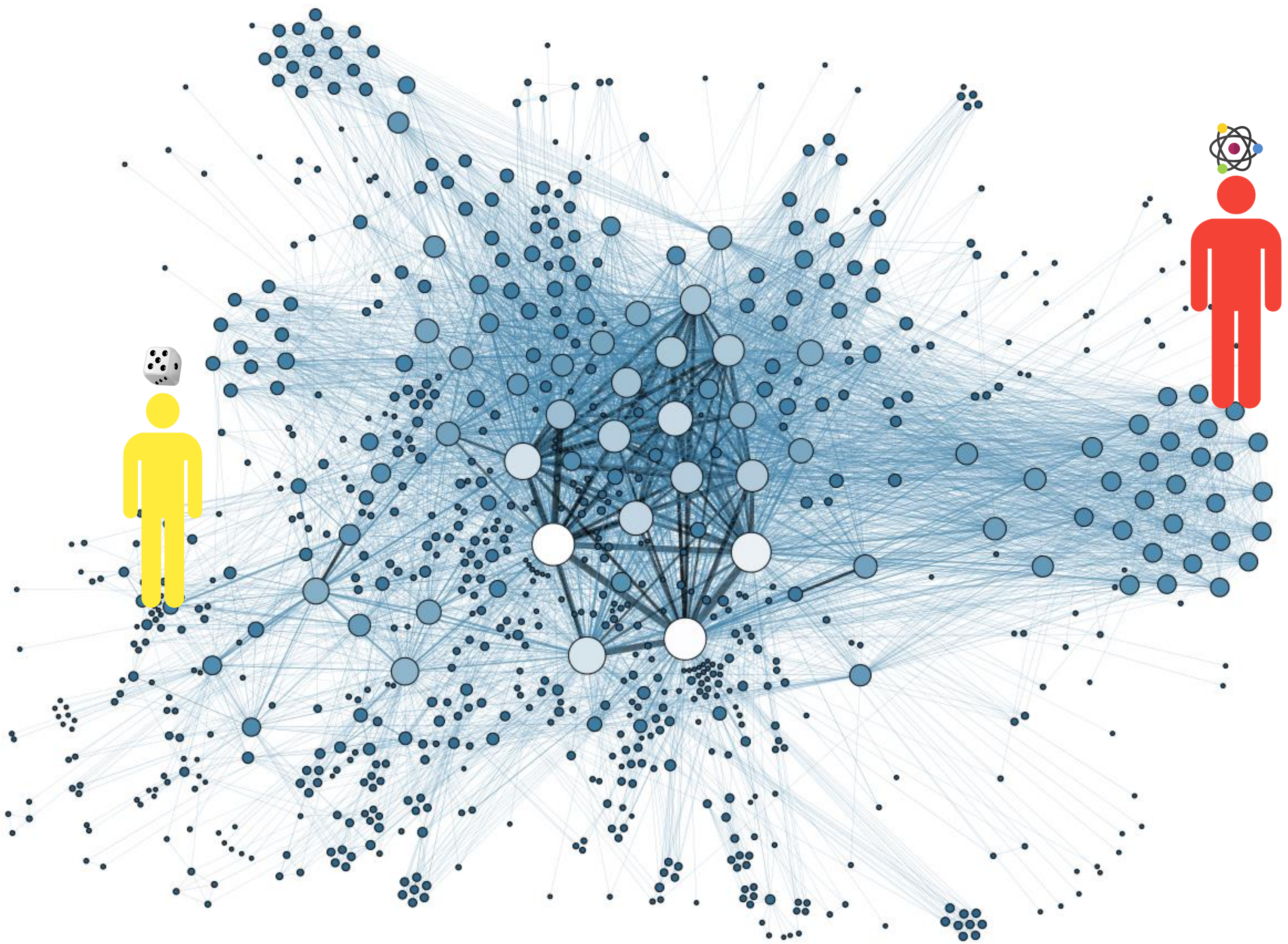
Laboratoire
Méthodes
Formelles

Random Walks & Quantum Walks

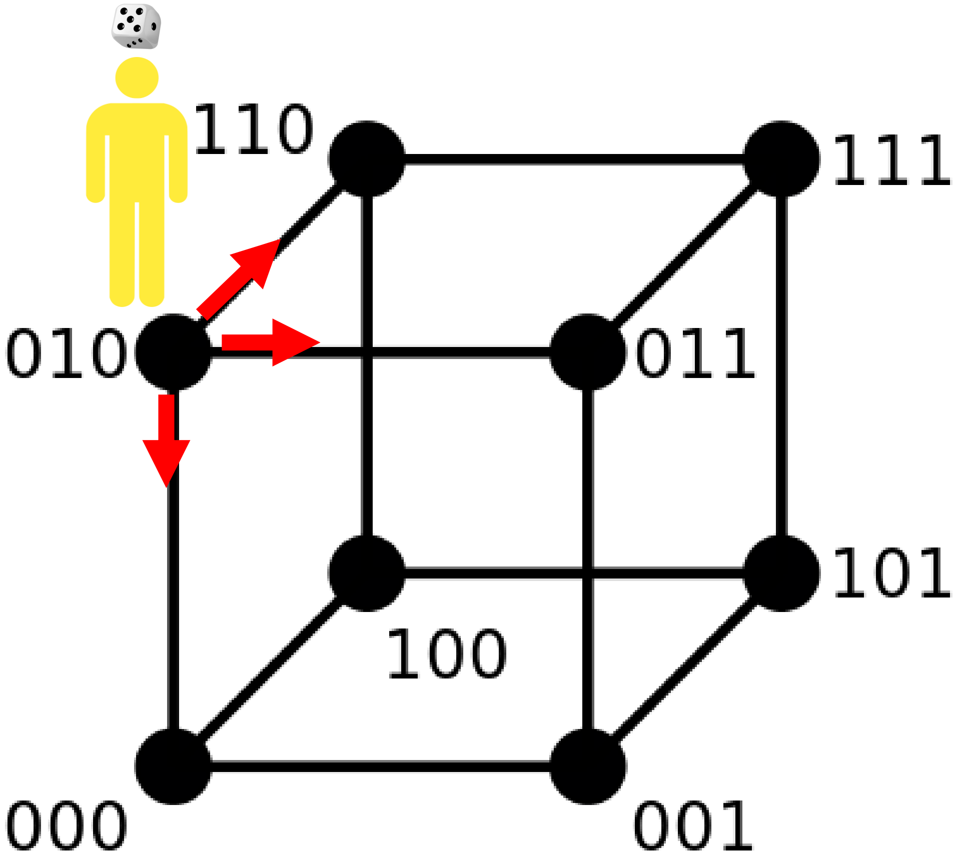
Marin Costes

Tuesday, 26^{th} November

Walking on a graph



Why would you walk randomly on a graph ?



- Some problems are simpler to solve using randomness.
- You save ressources using a random walk.

  Example [cube] :

  3 random bits → **1** random bit

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Stochastic process

- States :

$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_\alpha \end{pmatrix}$$

- Stochastic matrices :

$$\sum_{i=1}^{\alpha} P_{i,j} = 1$$

- Evolution :

$$P\, p^0 = p^1$$

$$P = \begin{bmatrix} P_{1,1} & P_{1,2} & \cdots & P_{1,j} & \cdots & P_{1,\alpha} \\ P_{2,1} & P_{2,2} & \cdots & P_{2,j} & \cdots & P_{2,\alpha} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ P_{i,1} & P_{i,2} & \cdots & P_{i,j} & \cdots & P_{i,\alpha} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ P_{\alpha,1} & P_{\alpha,2} & \cdots & P_{\alpha,j} & \cdots & P_{\alpha,\alpha} \end{bmatrix}.$$

Flip a coin

States :

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Transitions :

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$\downarrow P$

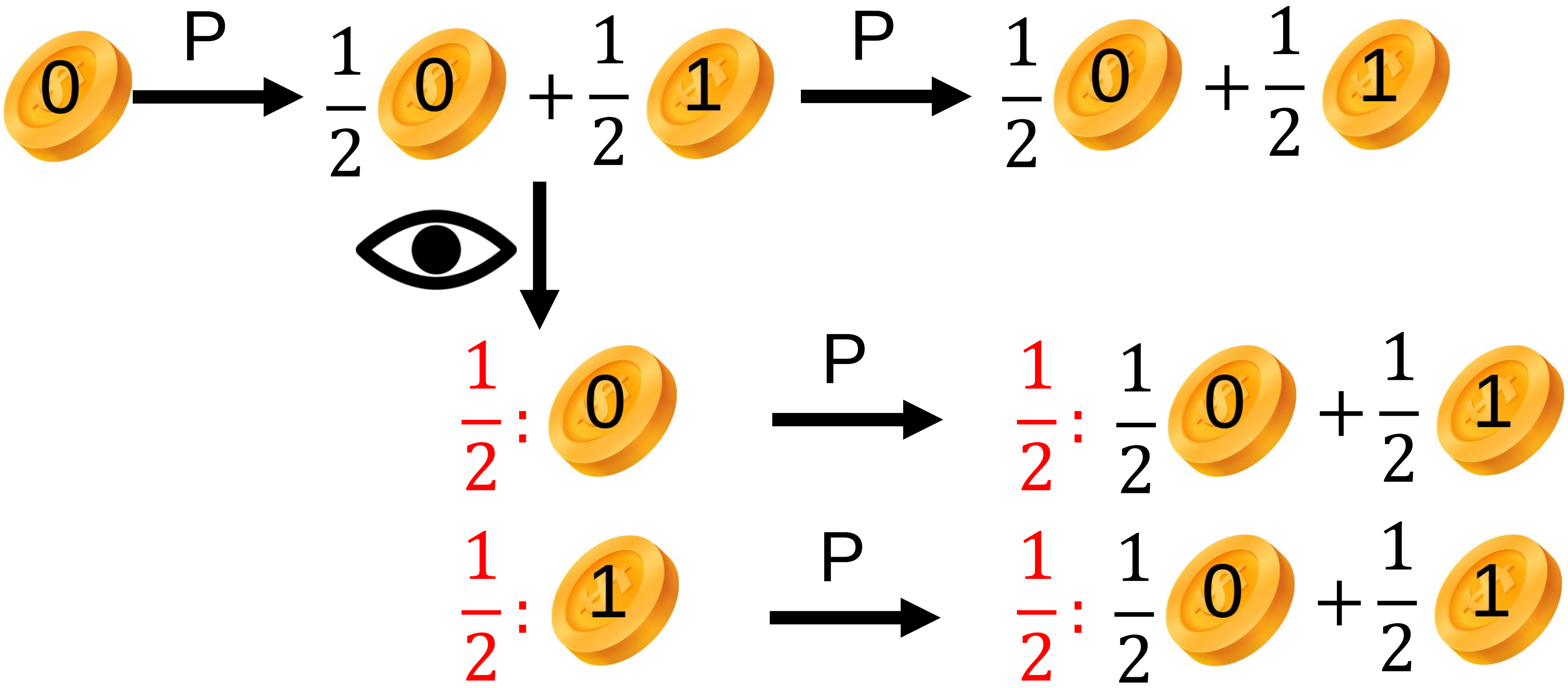
$0{,}5$

$+ \ 0{,}5$

$$= P \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0{,}5 \\ 0{,}5 \end{pmatrix}$$

Measurement

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$



Measurement doesn’t disturb the system !

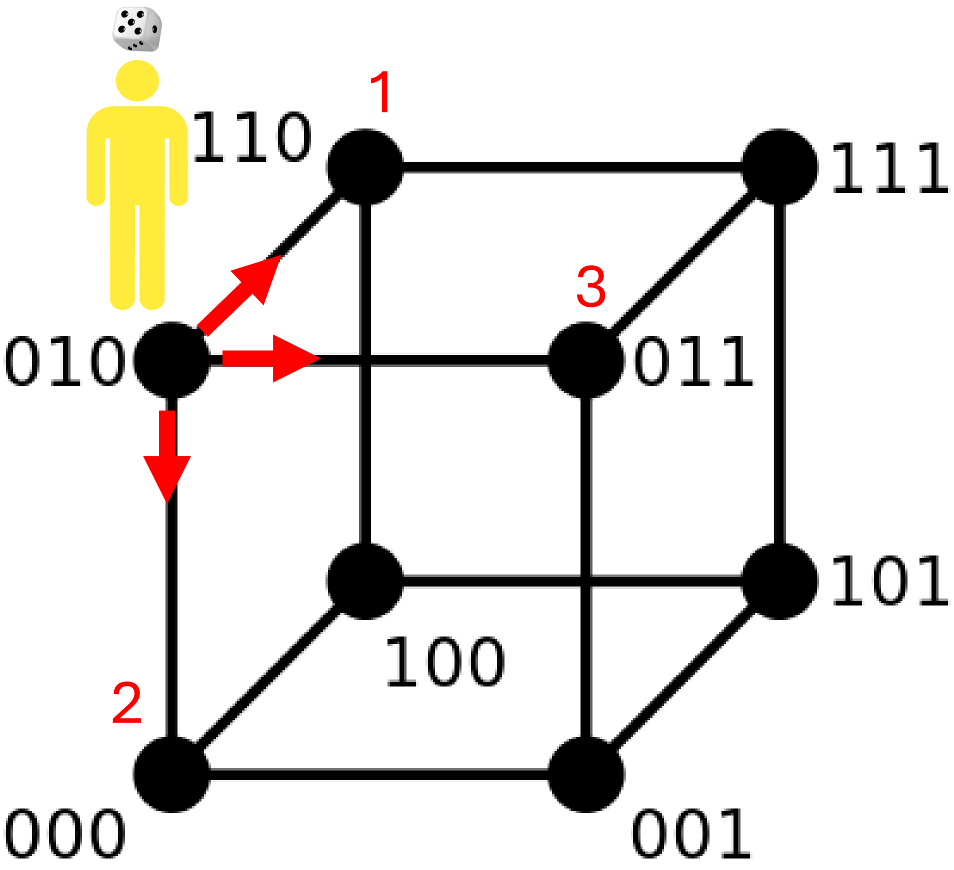
Random walks

- Usually states are vectors in V_{nodes}.

  Exemple : $|010>$
- Here we take vectors in $V_{nodes} \times V_{coin}$.

  Exemple : $|010>|2\rangle$
- First we toss the coin.

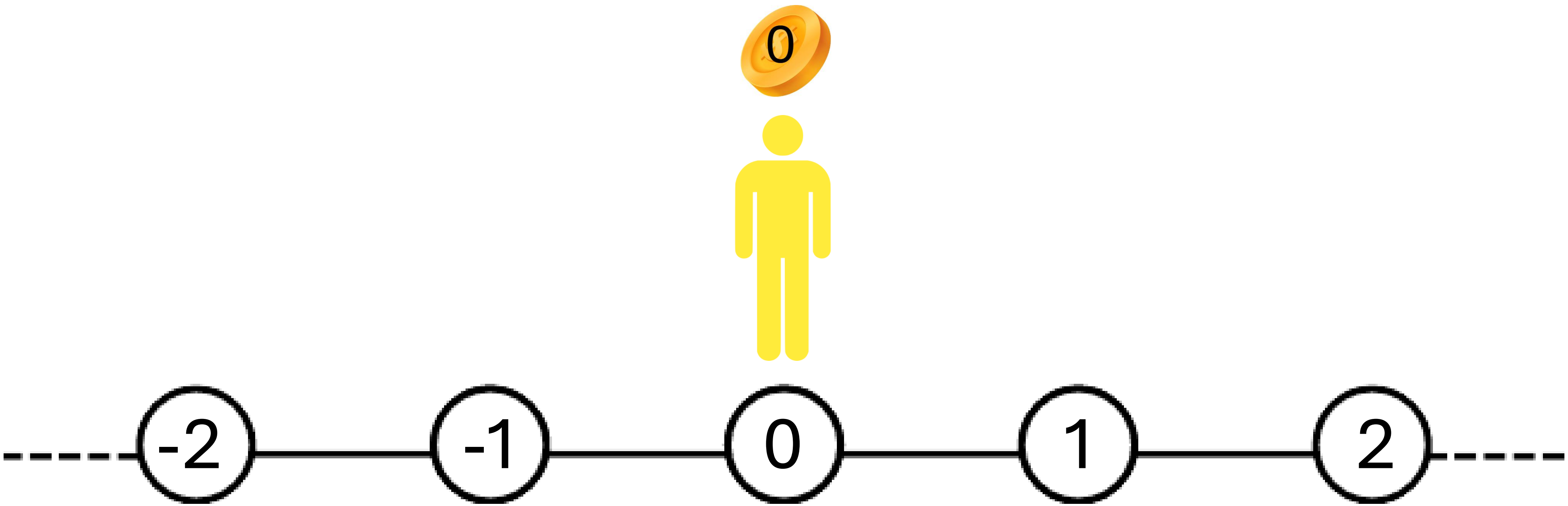
  Exemple : $|010>\sum_{i=1}^{3}\frac{1}{3}|i\rangle$
- Then we walk towards the node indicated by the coin.



Walking randomly on a line

State : $|0\rangle|0\rangle$

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

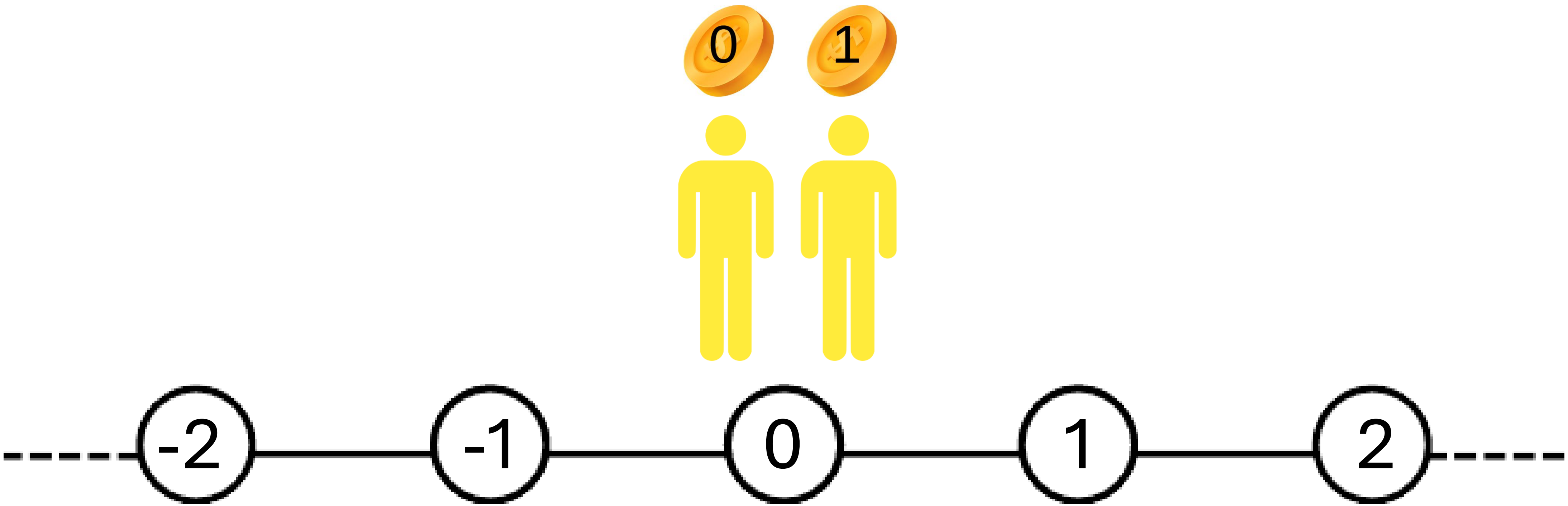


Probabilities : 1

Walking randomly on a line

State : $\color{red}{0{,}5}|0\rangle|0\rangle + \color{red}{0{,}5}|0\rangle|1\rangle$

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

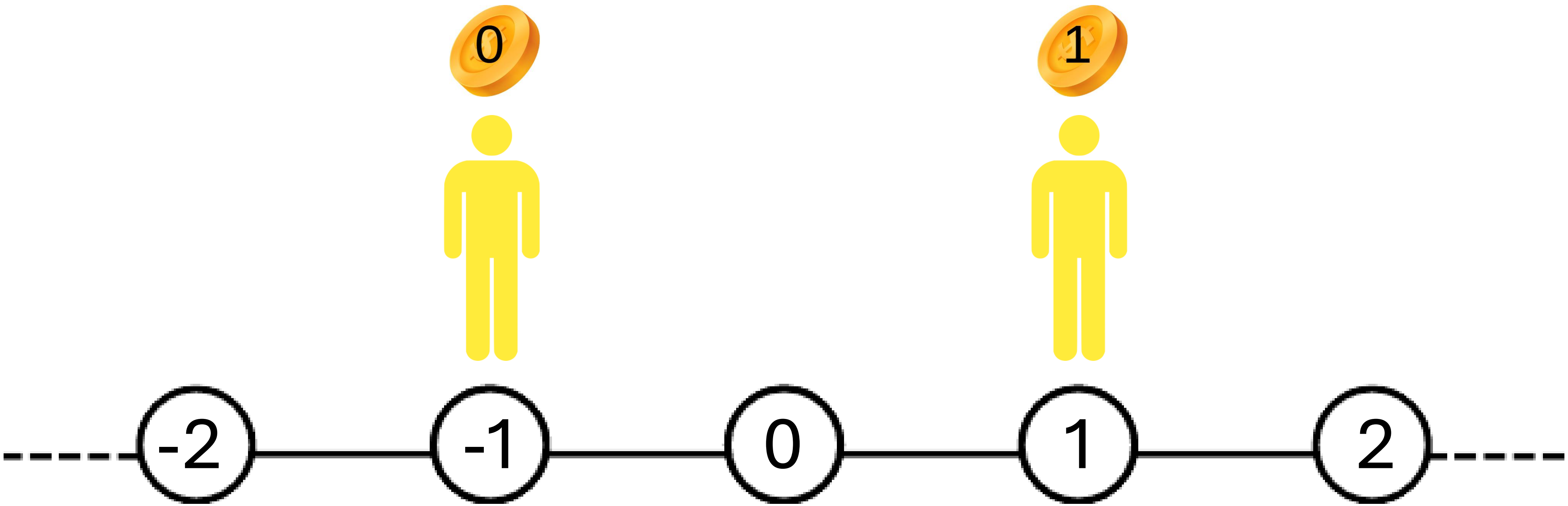


Probabilities : 1

Walking randomly on a line

State : $0{,}5|-1\rangle|0\rangle + 0{,}5|1\rangle|1\rangle$

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

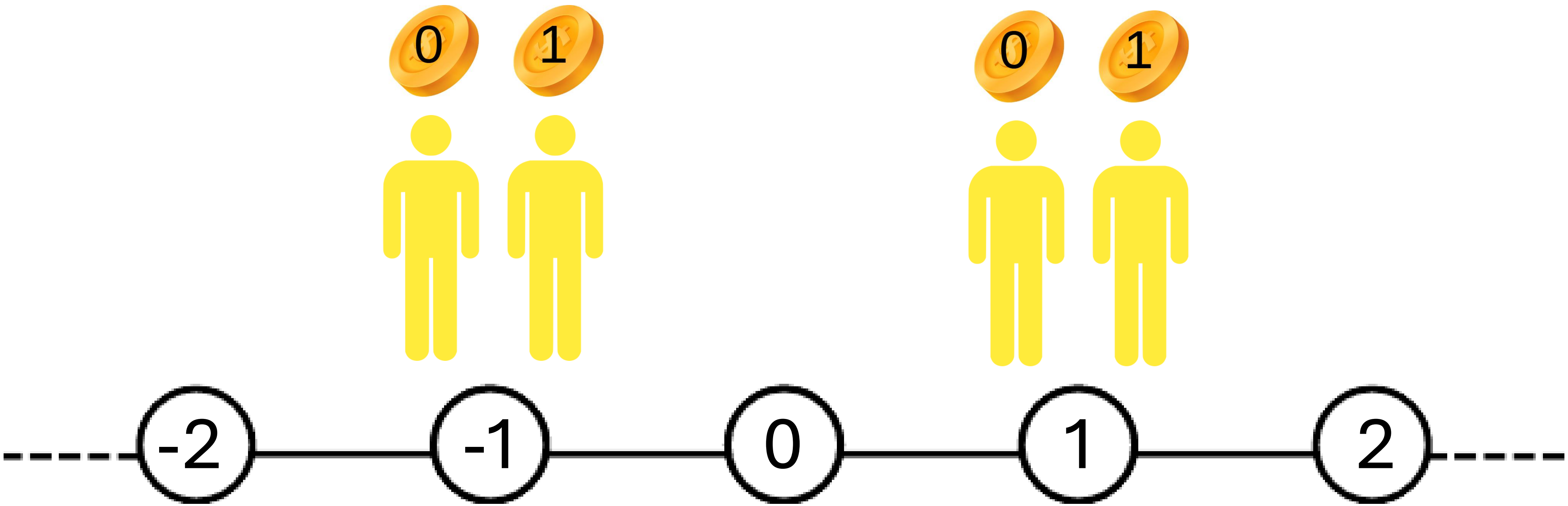


Probabilities : **0,5** **0,5**

Walking randomly on a line

State : $0{,}25|-1\rangle|0\rangle + 0{,}25|-1\rangle|1\rangle + 0{,}25|1\rangle|0\rangle + 0{,}25|1\rangle|1\rangle$

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

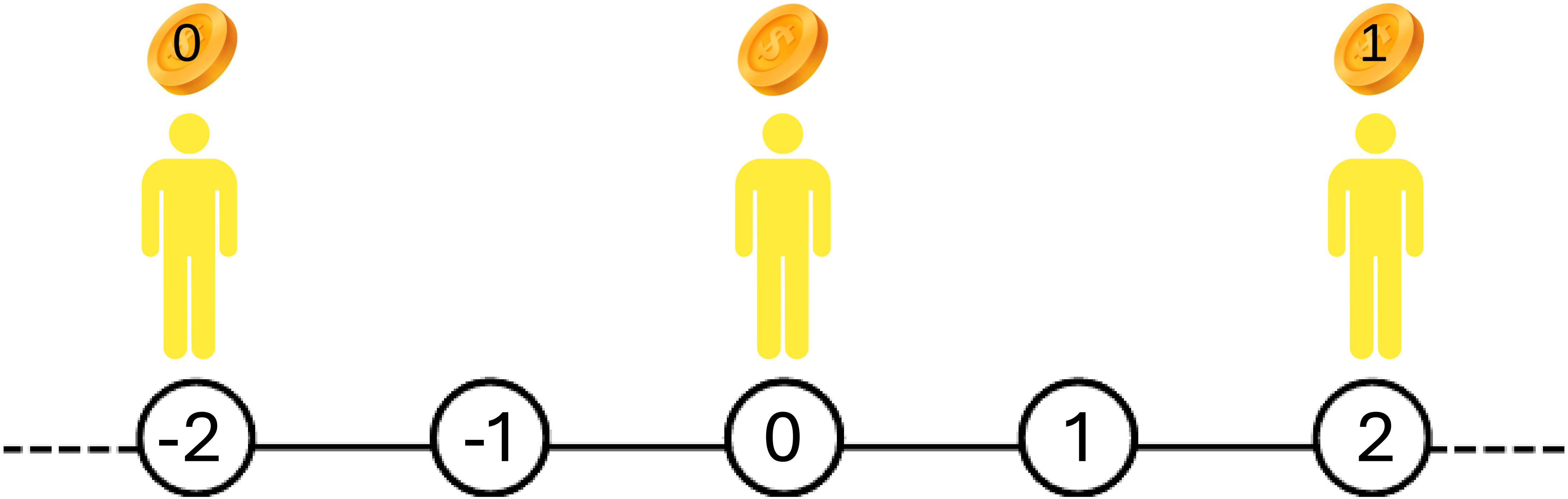


Probabilities : **0,5** **0,5**

Walking randomly on a line

State : $\color{red}{0{,}25}|-2\rangle|0\rangle + \color{red}{0{,}25}|2\rangle|1\rangle$
$+\color{red}{0{,}5}|0\rangle(\color{red}{0{,}5}|1\rangle + \color{red}{0{,}5}|0\rangle)$

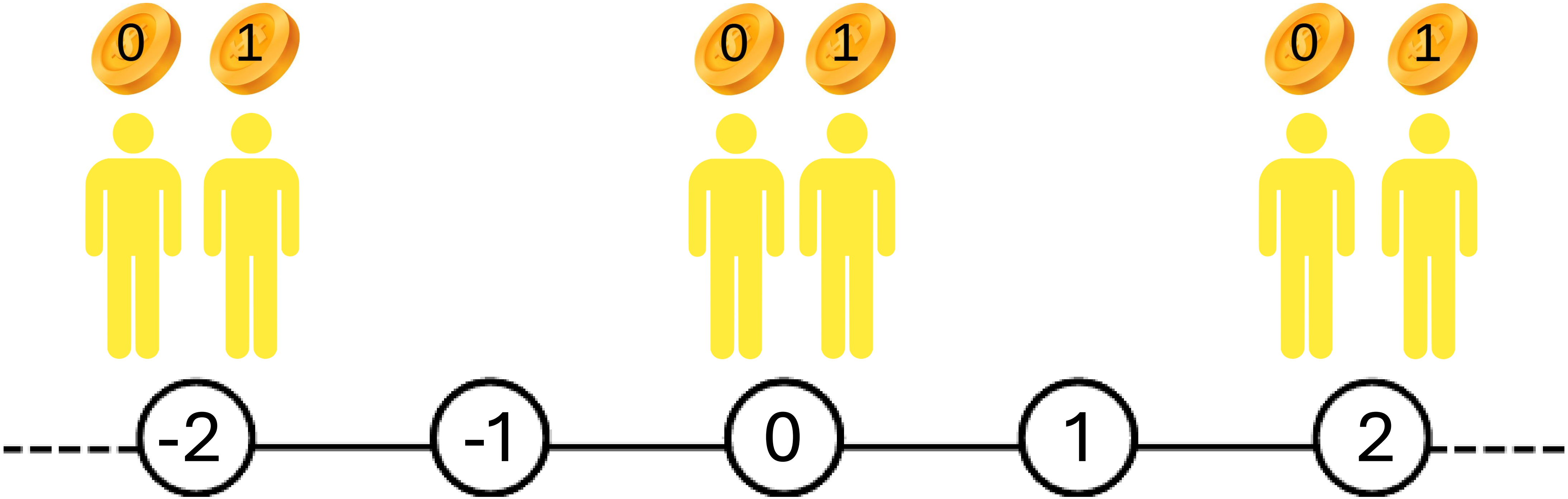
$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$



Probabilities : **0,25** **0,5** **0,25**

Walking randomly on a line

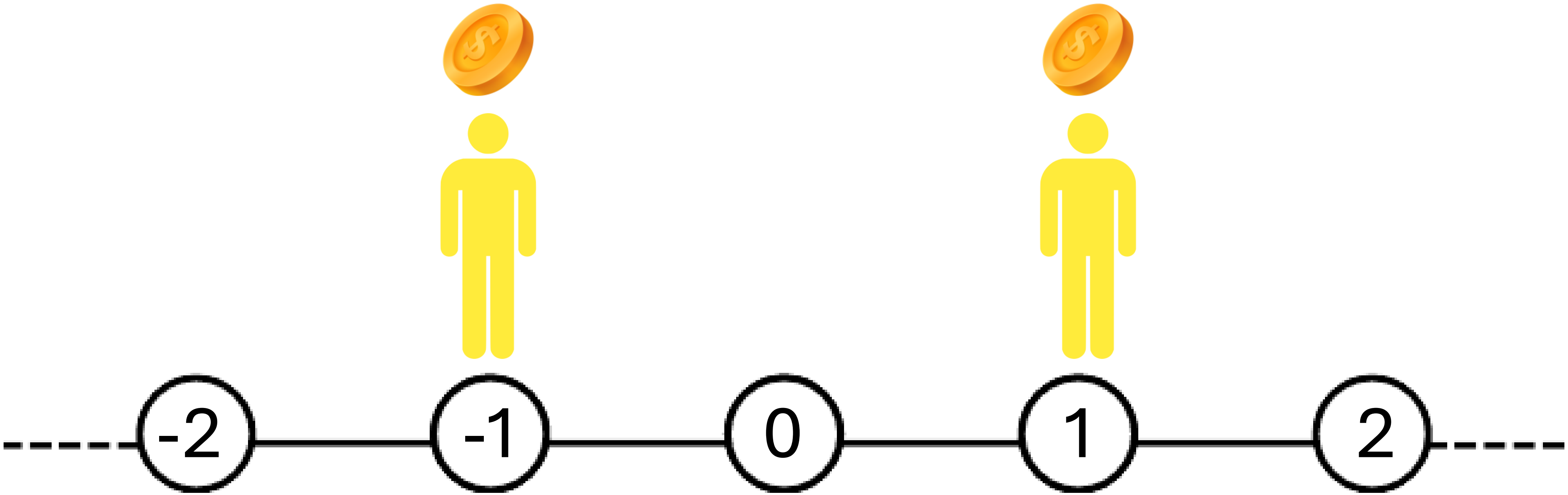
$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$



Probabilities : **0,25** **0,5** **0,25**

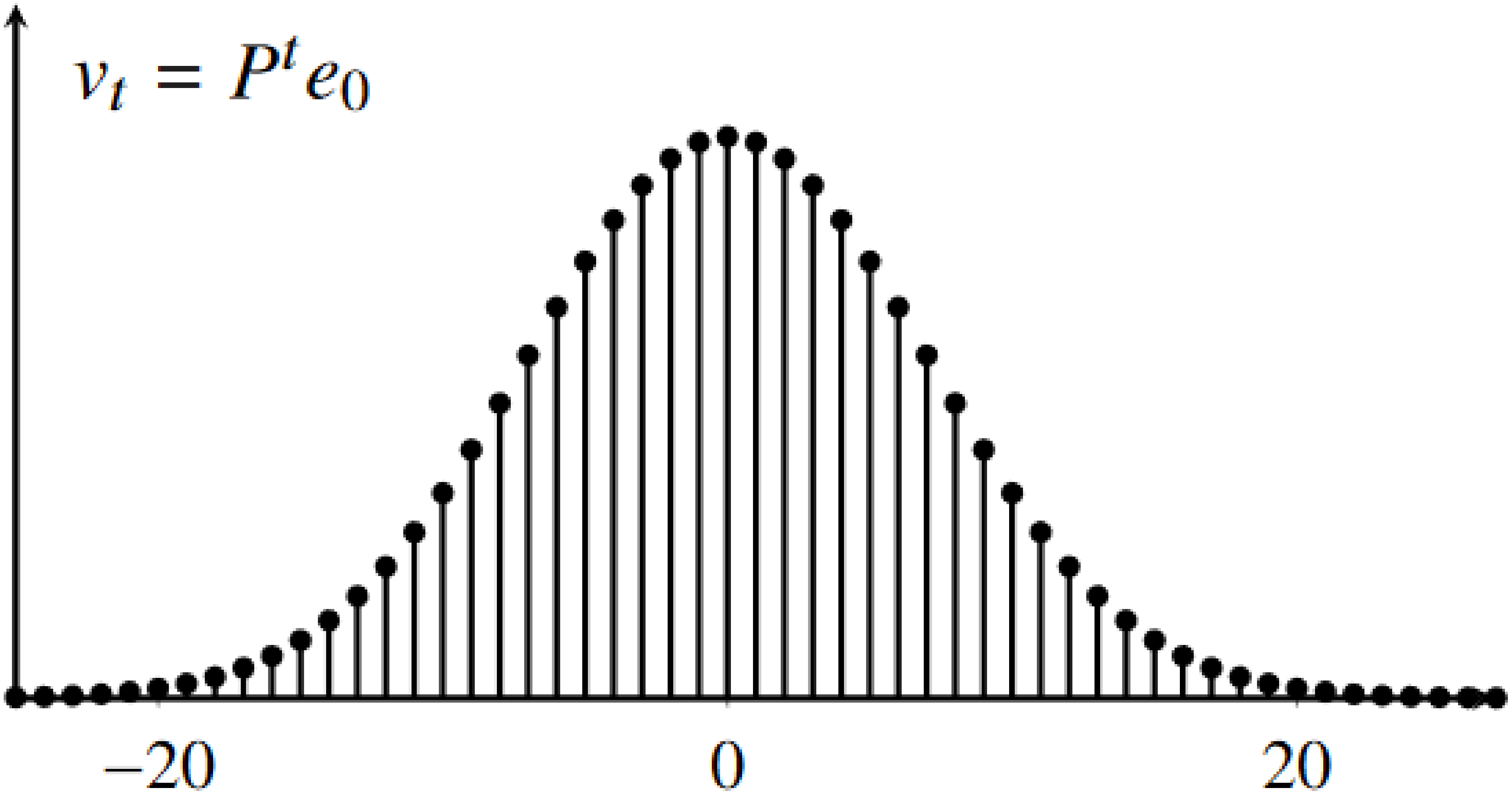
Walking randomly on a line

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$



Probabilities : **0,375** **0,375**

After many steps...



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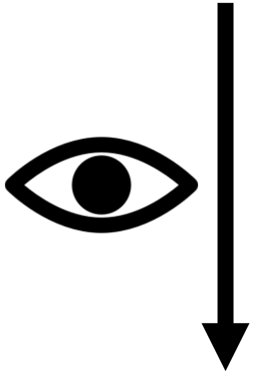
Quantum states

 : $p = p_0|0\rangle + \cdots + p_\alpha|\alpha\rangle = \begin{pmatrix} p_0 \\ \vdots \\ p_\alpha \end{pmatrix}$ with $p_i \in \mathbb{R}$ such that $\sum_{i=0}^{\alpha} p_i = 1$

 : $\mathrm{u} = u_0|0\rangle + \cdots + u_\alpha|\alpha\rangle = \begin{pmatrix} u_0 \\ \vdots \\ u_\alpha \end{pmatrix}$ with $u_i \in \mathbb{C}$ such that $\sum_{i=0}^{\alpha} \|u_i\|^2 = 1$

Observing quantum states

 $: \mathrm{u} = u_0|0\rangle + \cdots + u_\alpha|\alpha\rangle = \begin{pmatrix} u_0 \\ \vdots \\ u_\alpha \end{pmatrix}$ with $u_i \in \mathbb{C}$ such that $\sum_{i=0}^{\alpha} \|u_i\|^2 = 1$



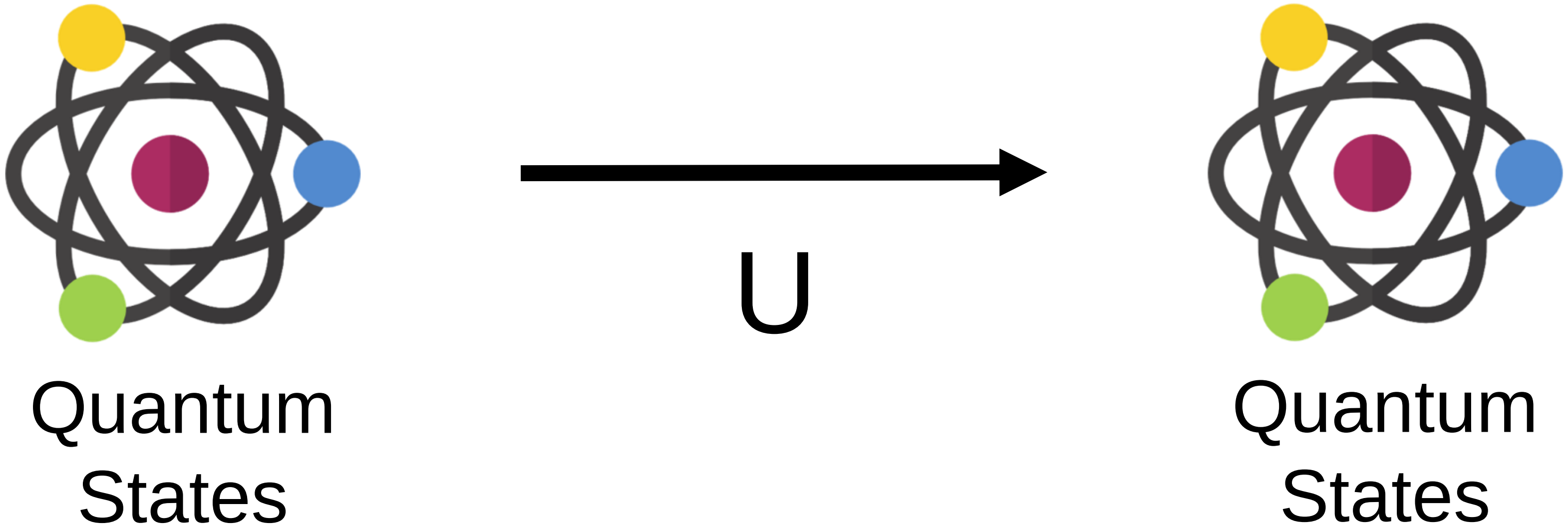
$|0\rangle$ with probability $\|u_0\|^2$

$\vdots$

$|\alpha\rangle$ with probability $\|u_\alpha\|^2$

Quantum operations

Unitary matrix U : $\|Uv\|_2 = \|v\|_2$



Flip a quantum coin

States :

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Transitions :

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$\downarrow U$

$\frac{1}{\sqrt{2}}$


$+ \frac{1}{\sqrt{2}}$


$\downarrow U$

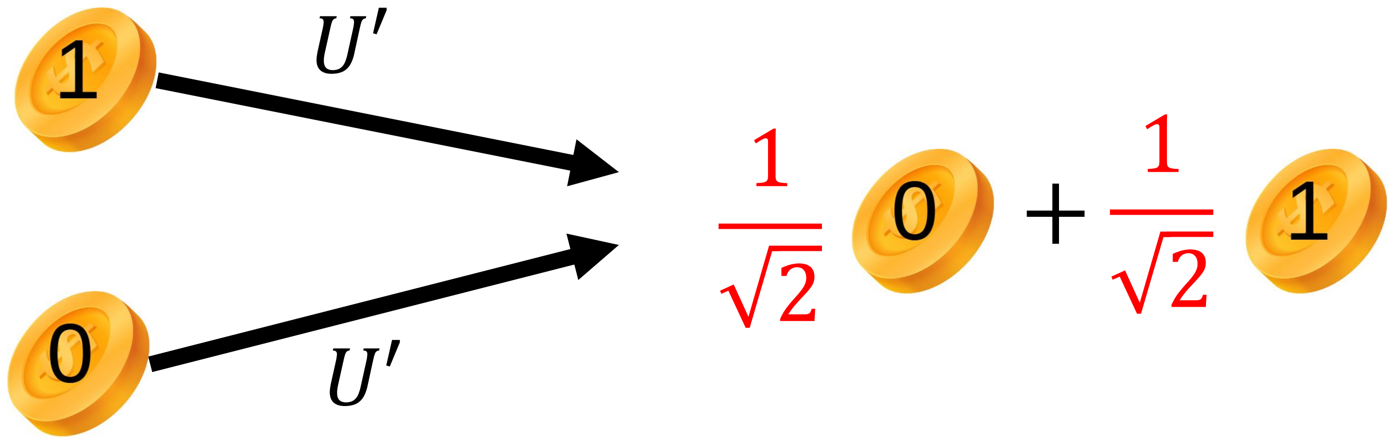
$\frac{1}{\sqrt{2}}$


$- \frac{1}{\sqrt{2}}$


Reversibility

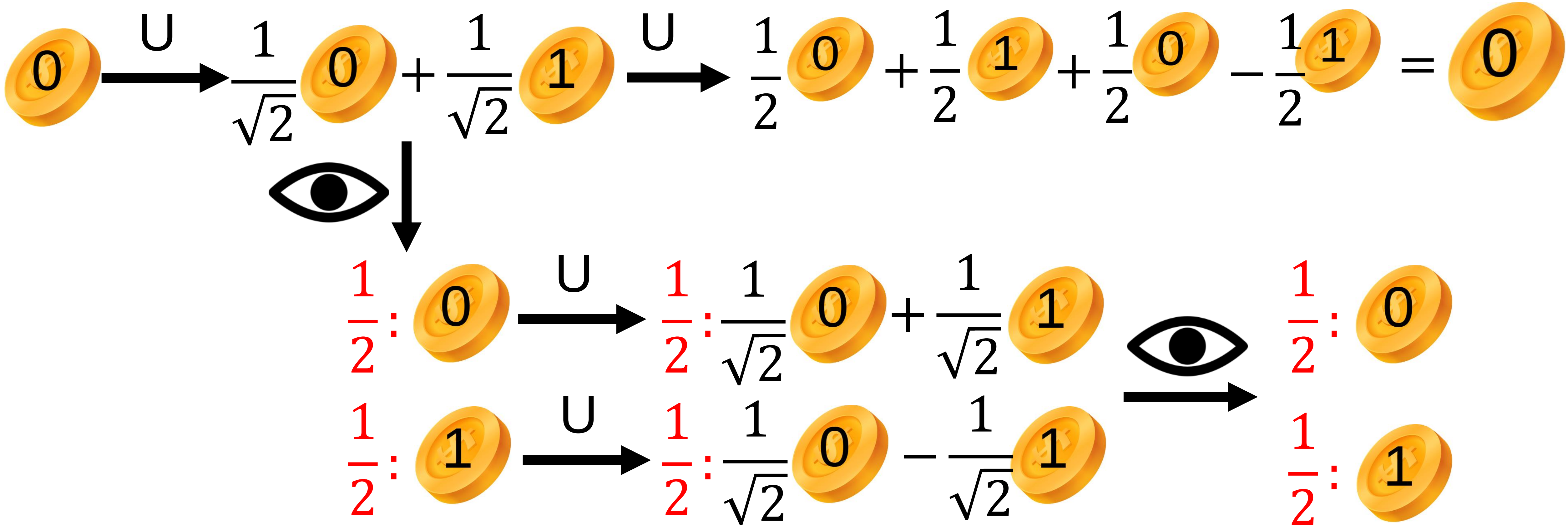
All unitary matrices are inversible !

$$U' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$ is not inversible (and therefore not unitary).



Destructive measurement

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

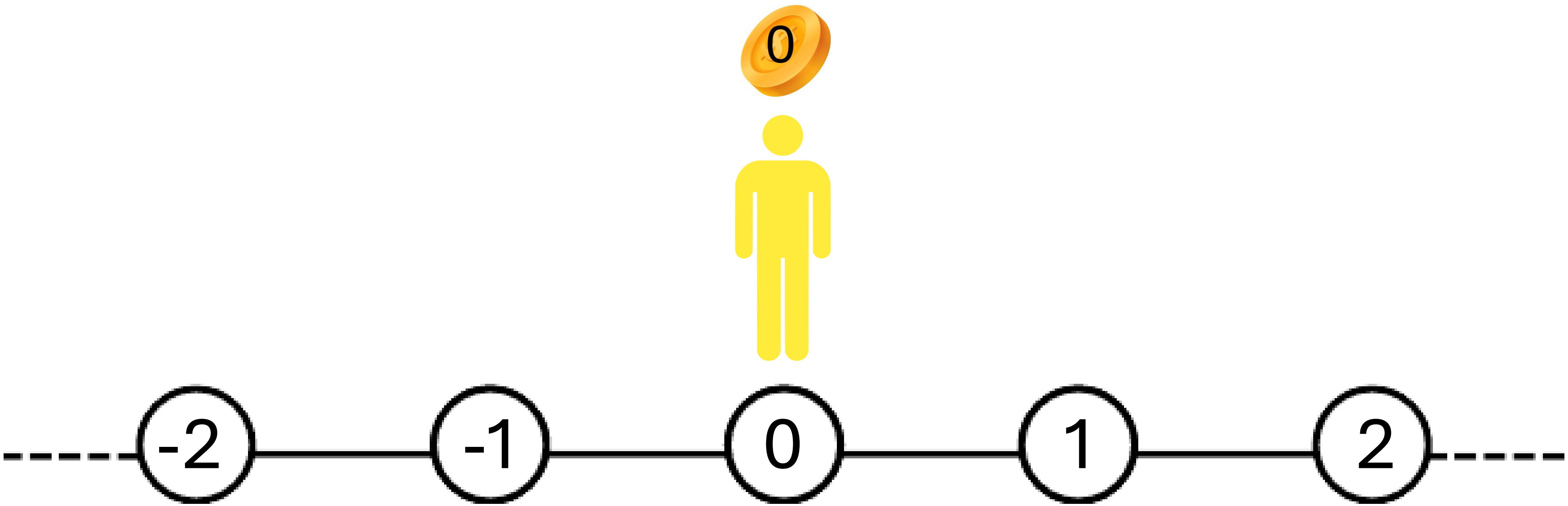


Measurement disturbs the system !

Quantum Walk on a line

State : $|0\rangle|0\rangle$

$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

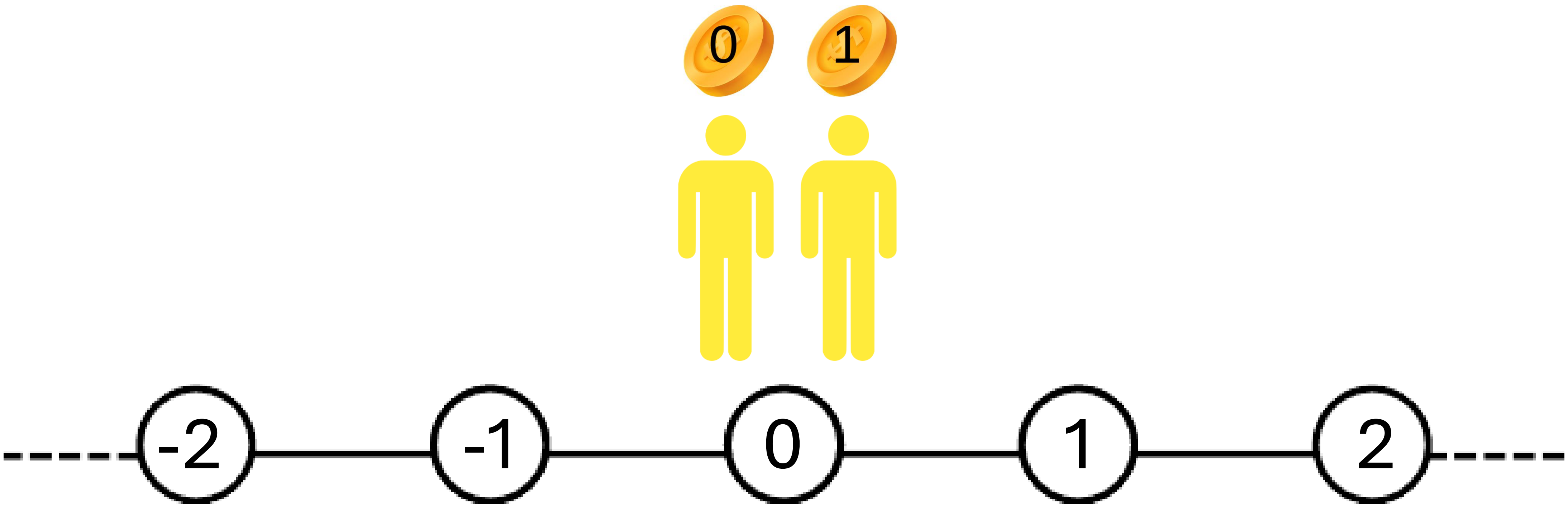


Probabilities : 1

Quantum Walk on a line

State : $\color{red}{1/\sqrt{2}}|0\rangle|0\rangle + \color{red}{1/\sqrt{2}}|0\rangle|1\rangle$

$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

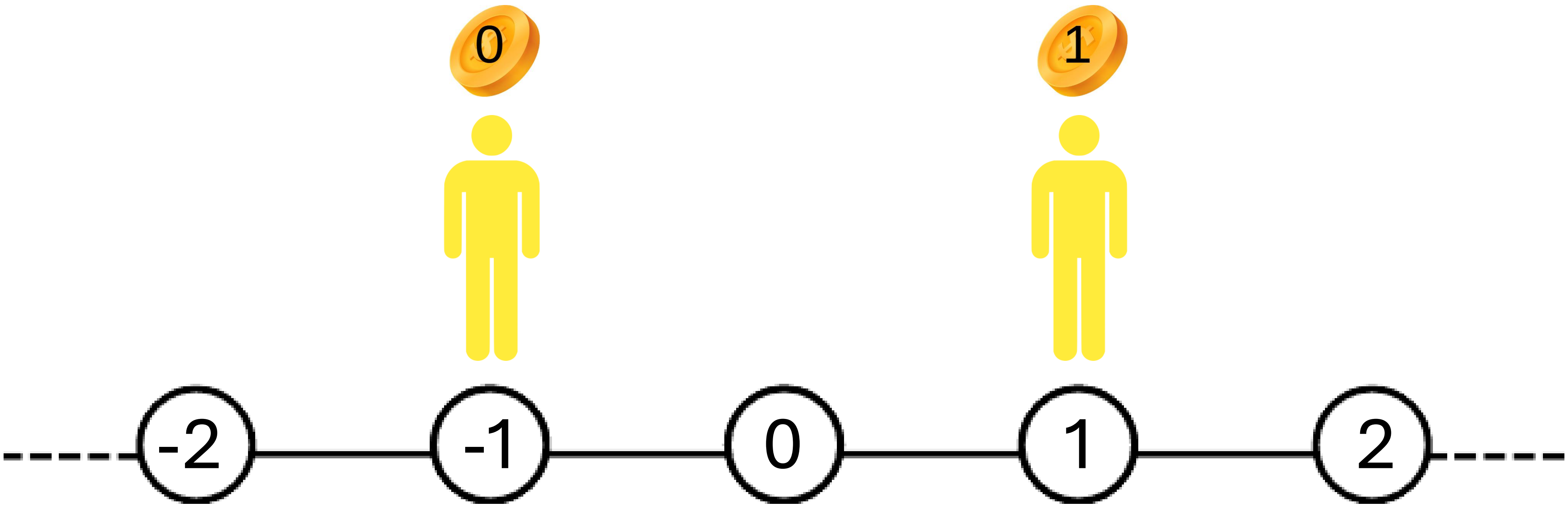


Probabilities : **1**

Quantum Walk on a line

State : $1/\sqrt{2}|-1\rangle|0\rangle + 1/\sqrt{2}|1\rangle|1\rangle$

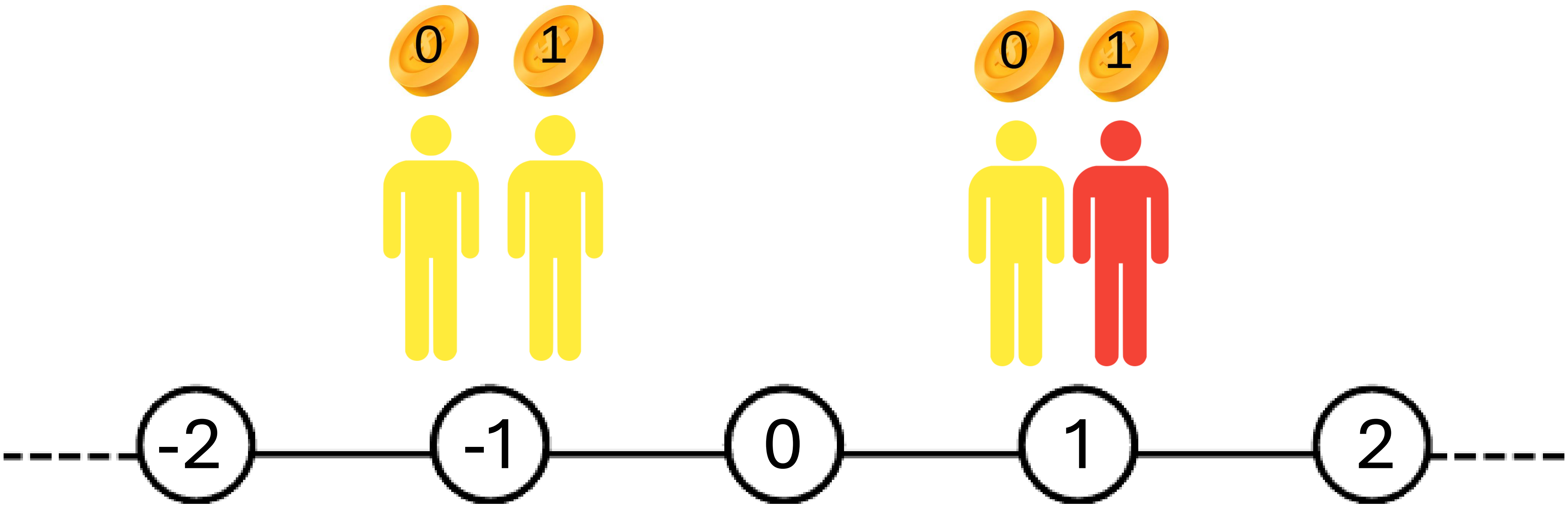
$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$



Probabilities : **0,5** **0,5**

Quantum Walk on a line

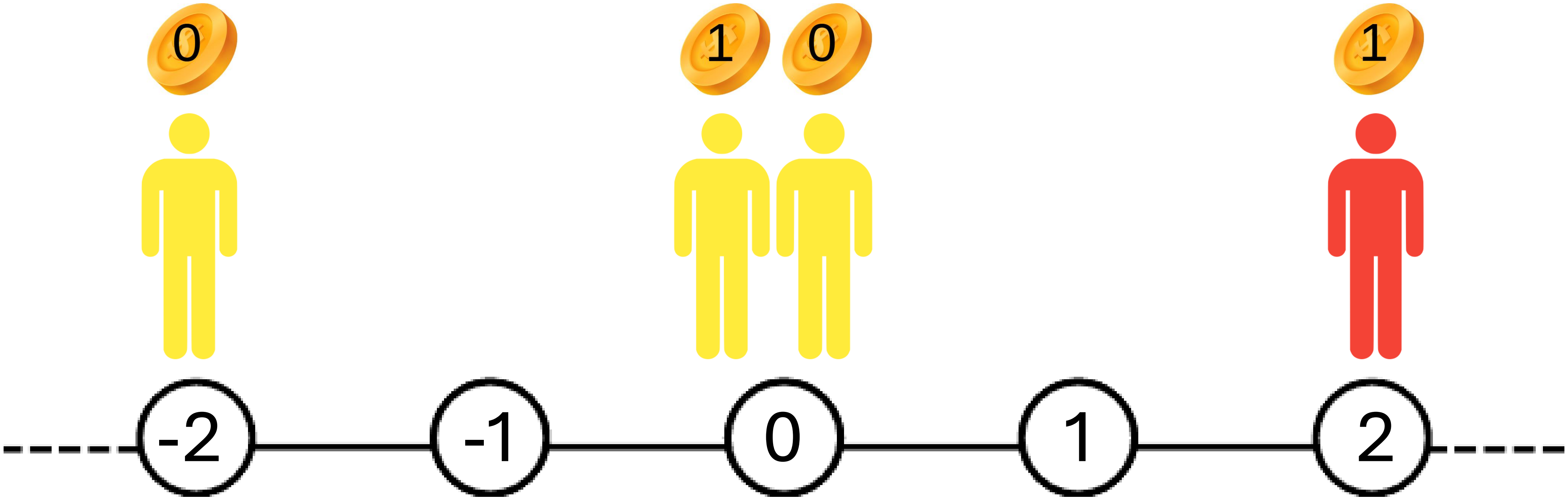
$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$



Probabilities : **0,5** **0,5**

Quantum Walk on a line

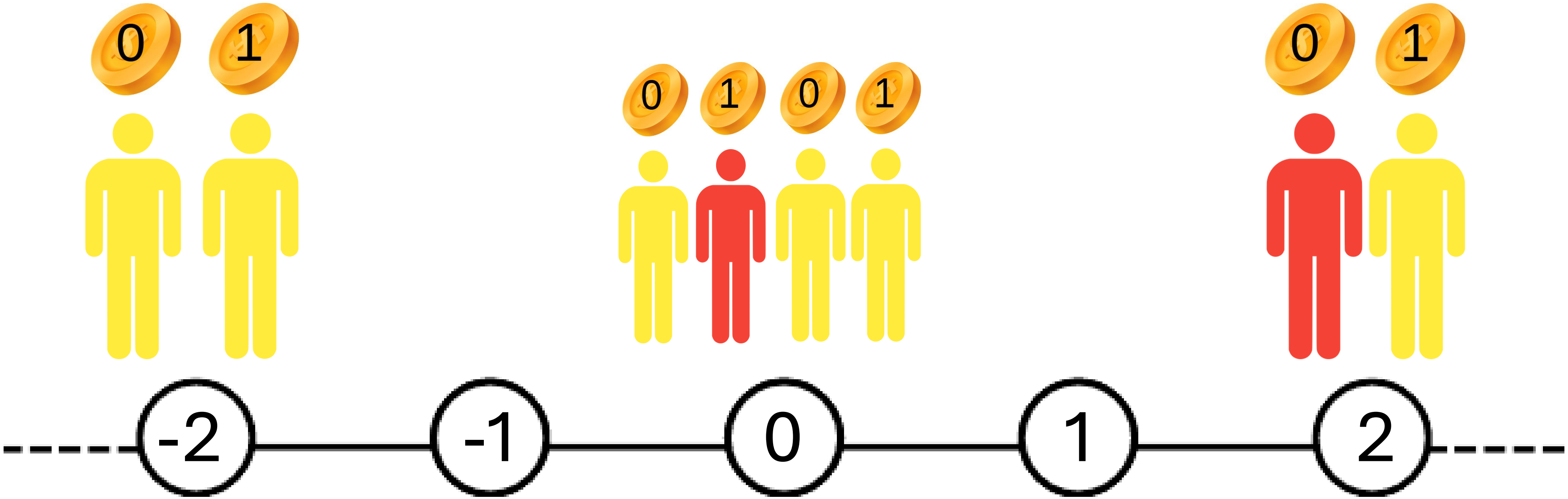
$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$



Probabilities : **0,25** **0,5** **0,25**

Quantum Walk on a line

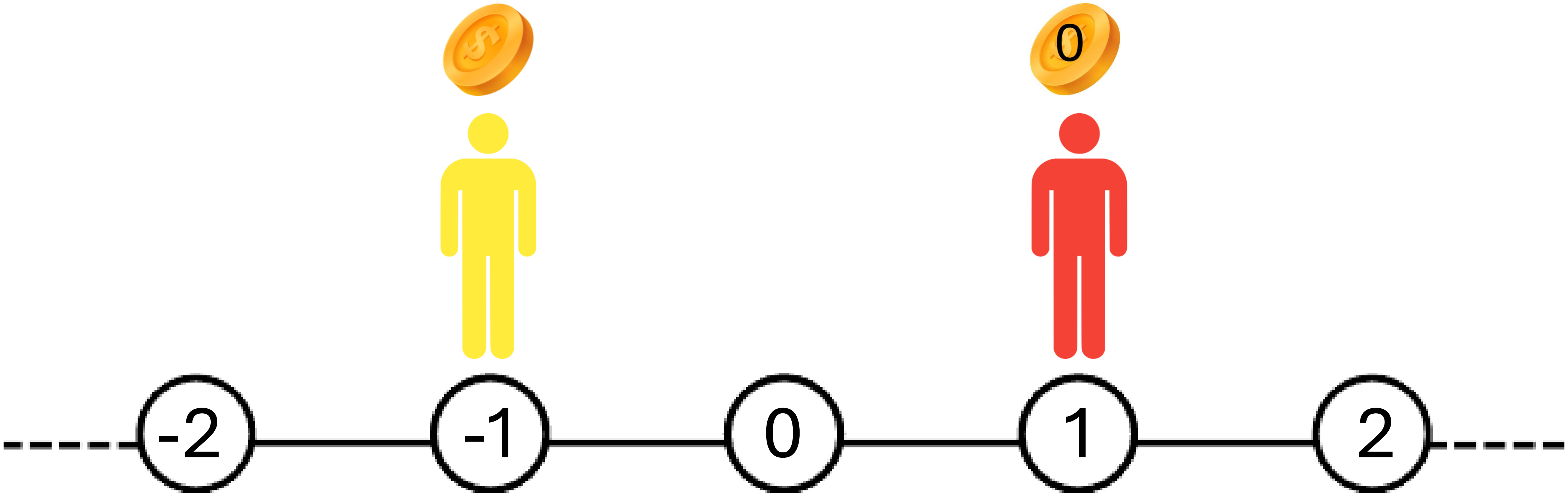
$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$



Probabilities : **0,25** **0,5** **0,25**

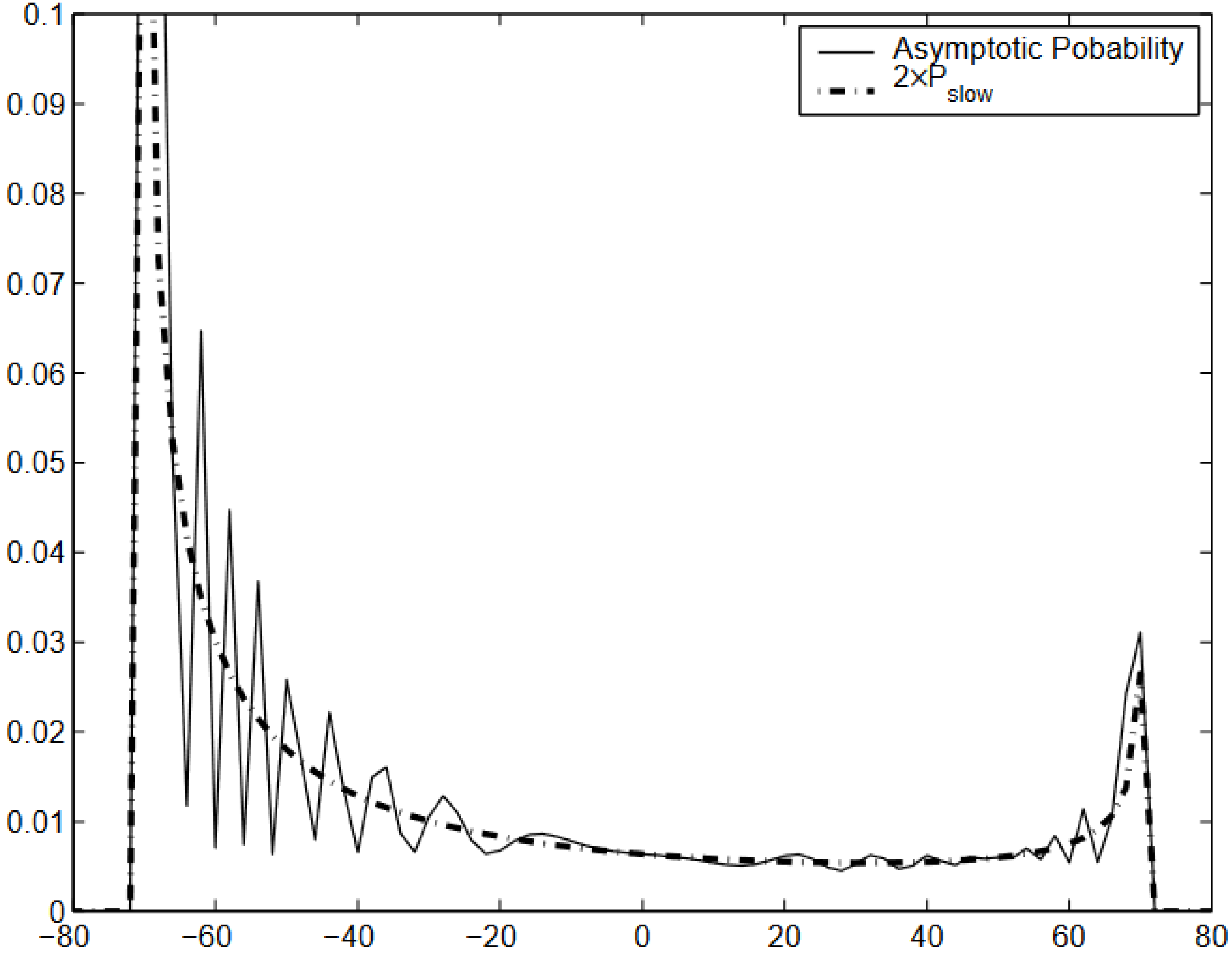
Quantum Walk on a line

$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

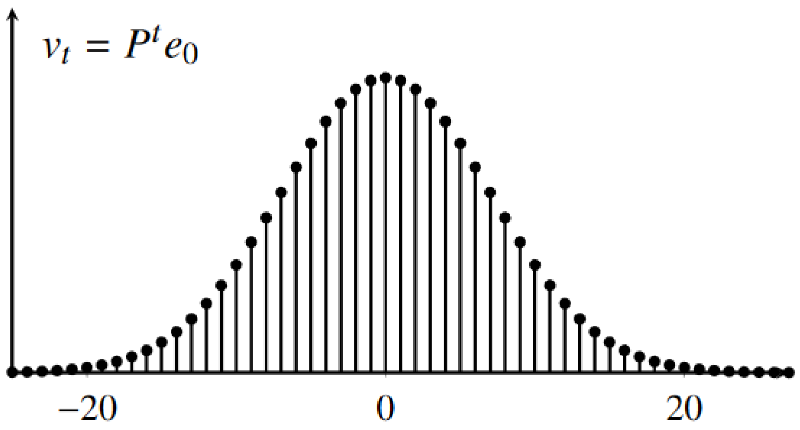


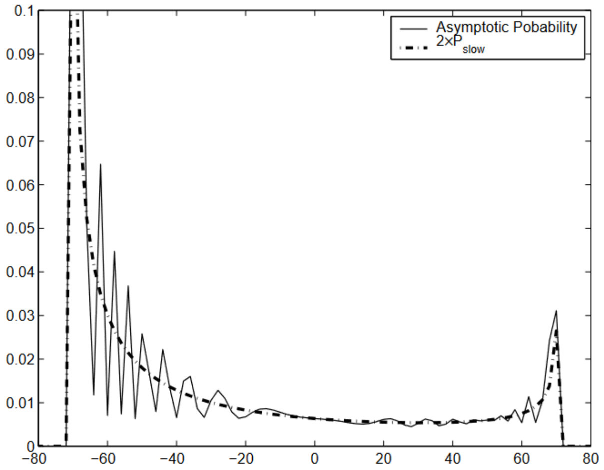
Probabilities : **0,625** **0, 125**

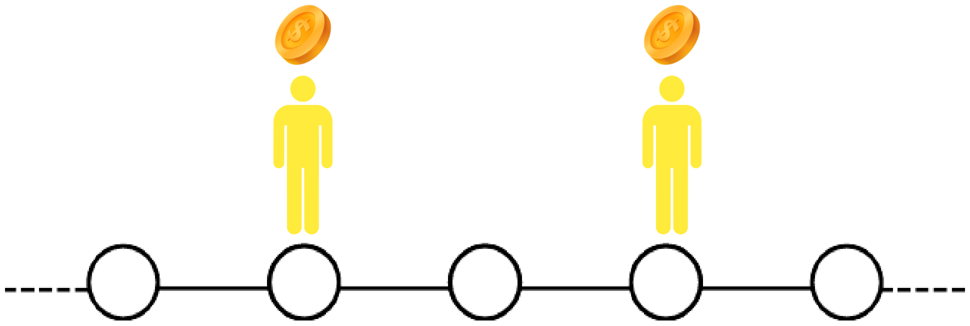
After many steps…



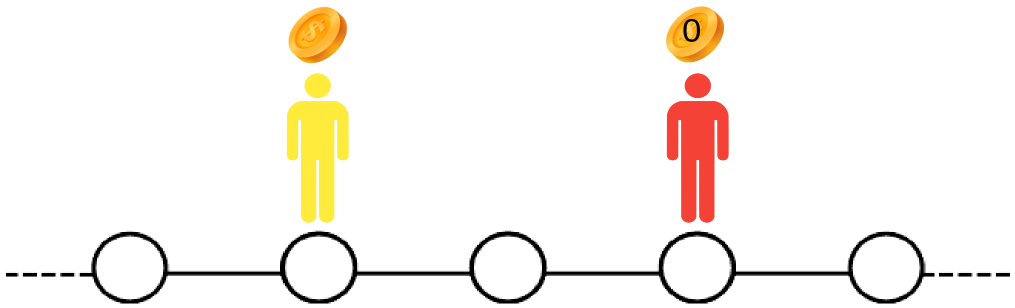
Random VS Quantum







0,375 0,375



0,625 0, 125

Random VS Quantum

- Can implement irreversible dynamics ;
- Non destructive measurement ;
- Doesn't feature interferences.

- Can only implement reversible dynamics ;
- Destructive measurement ;
- Features interferences.

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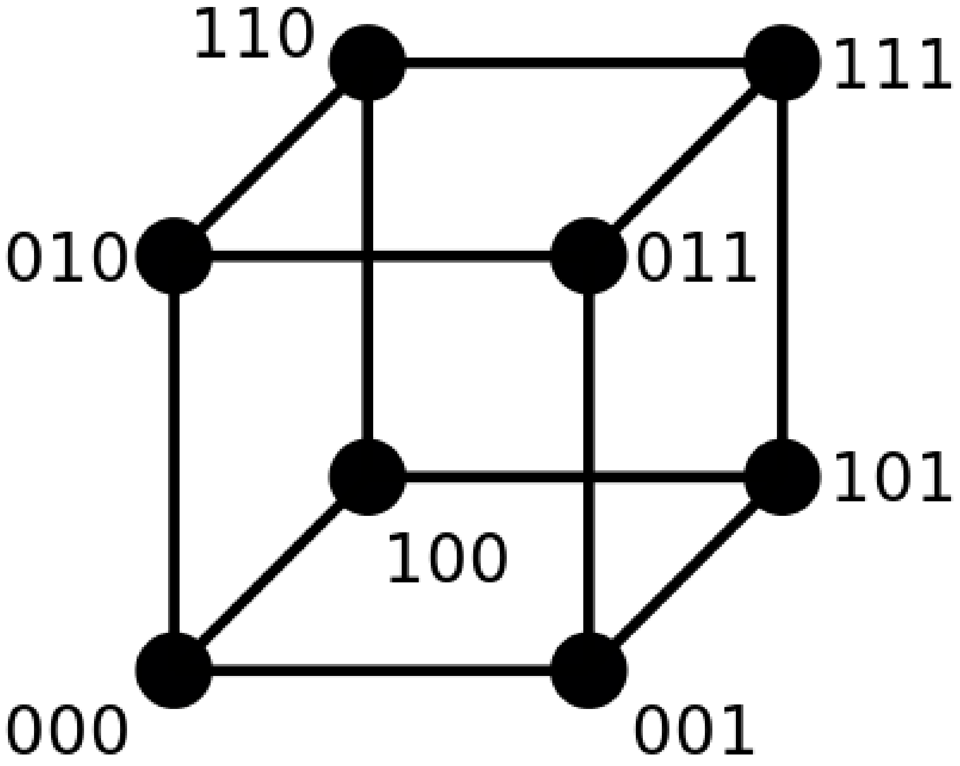
Local search problem

- We search in a graph some marked elements. Exemple : 110
- We can only use a random walk P on this graph. Exemple : draw one of your neighbours at random.
- How long does it takes ?



Two settings

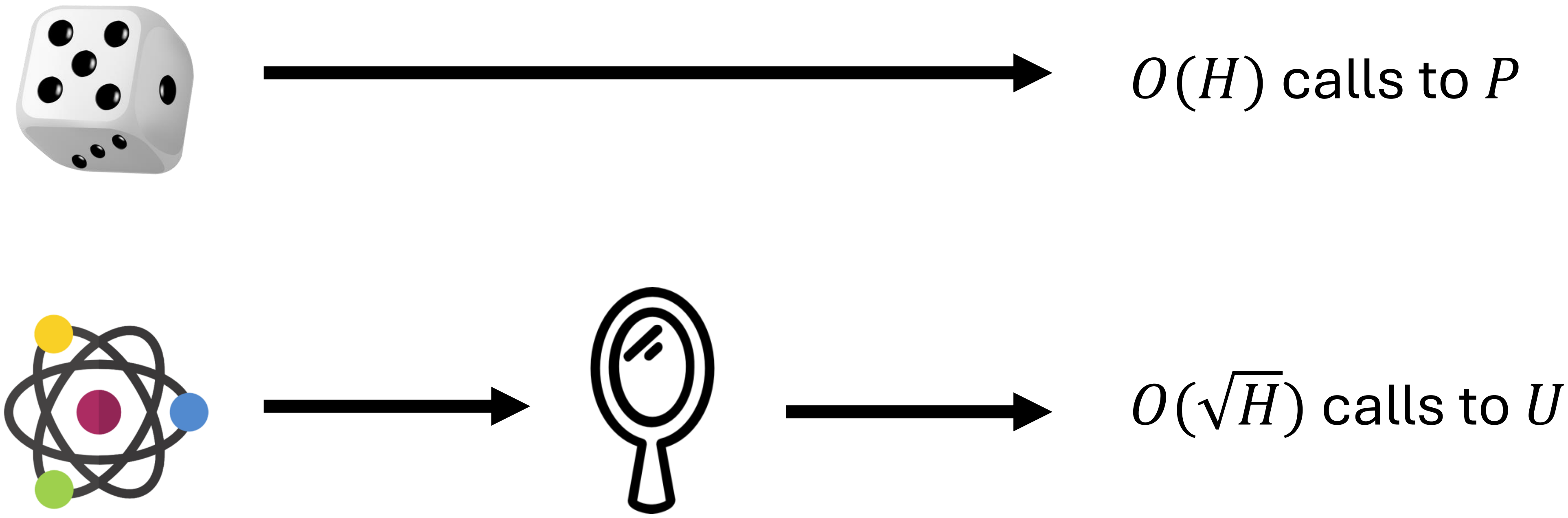
- We have access to a random walk (without coin) :

$$P = \begin{bmatrix} P_{1,1} & P_{1,2} & \dots & P_{1,j} & \dots & P_{1,\alpha} \\ P_{2,1} & P_{2,2} & \dots & P_{2,j} & \dots & P_{2,\alpha} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ P_{i,1} & P_{i,2} & \dots & P_{i,j} & \dots & P_{i,\alpha} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ P_{\alpha,1} & P_{\alpha,2} & \dots & P_{\alpha,j} & \dots & P_{\alpha,\alpha} \end{bmatrix}.$$

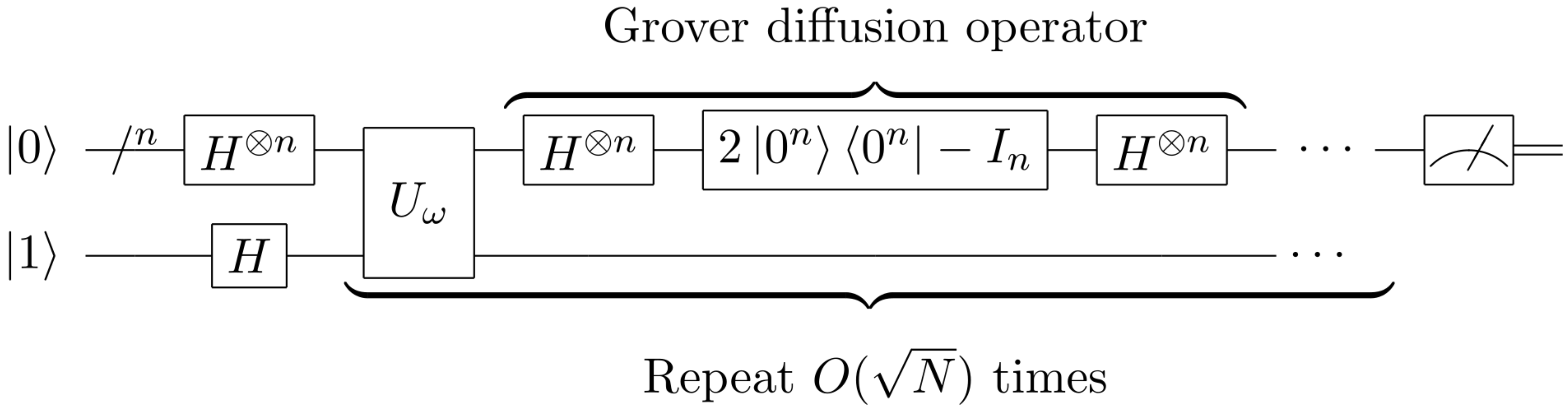
- We have access to the associated quantum coin toss operator :

$$U|i\rangle|0\rangle = |i\rangle \sum_j \sqrt{P_{i,j}}|j\rangle$$

Efficiencies



Groever Search



Find an element in a database quadratically faster than classicaly !

Thank you !

- Can implement irreversible dynamics ;
- Non destructive measurement ;
- Doesn’t feature interferences.

- Can only implement reversible dynamics ;
- Destructive measurement ;
- Features interferences **which allow to search quadratically faster.**

References

- Introduction to Quantum Walks:
  - https://arxiv.org/pdf/quant-ph/0303081
- Results about the quadratic speed up for the local search Pb:
  - https://www.irif.fr/~magniez/PAPIERS/kmor-algorithmica16.pdf
- Knowing everything about Quantum Computation:
  - Nielsen, M. A., & Chuang, I. L. (2010). *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge: Cambridge University Press.
- Knowing everything about Random Walks:
  - https://pages.uoregon.edu/dlevin/MARKOV/markovmixing.pdf