Generic Bidirectional Typing for Dependent Type Theories

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LMF Non-Permanent Members Seminar

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Dependent type theory, in a nutshell

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- Terms have types

$$\Gamma \vdash [0, 1, 2] : \mathsf{List\ Nat}$$

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$$\Gamma \vdash \lambda n.[1, \ldots, n] : \mathsf{Nat} \rightarrow \mathsf{List\ Nat}$$

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$$\Gamma \vdash \lambda n.[1, \ldots, n] : \Pi(n : \mathsf{Nat}).\mathsf{Vec\ Nat}\ n$$

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$$\Gamma \vdash \lambda n.[1, \ldots, n] : \Pi(n : \mathsf{Nat}).\mathsf{Vec}\ \mathsf{Nat}\ n$$

- Types are equal modulo computation

$$\frac{\Gamma \vdash [0, 1, 2] : \mathsf{Vec}\ \mathsf{Nat}\ 3 \qquad 2 + 1 \equiv 3}{\Gamma \vdash [0, 1, 2] : \mathsf{Vec}\ \mathsf{Nat}\ (2 + 1)}$$

Why dependent type theory?

Curry-Howard correspondence Deep link between type theory and logic

Propositions as types, proofs as programs. Proof/type theory dictionary

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Dependently-typed programming Dependent types allow to write both data and specification in the *same* language

(* pre-condition: list not empty *)

$\mathsf{hd} : \mathsf{List}\ \mathsf{Nat} \longrightarrow \mathsf{Nat}$

$\mathsf{hd} \quad (x :: l) = x$

$\mathsf{hd} \quad [] = \textbf{FAIL}$

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Type annotations in dependent type theory

In its most primitive form, syntax is extremely annotated and verbose

$$\frac{\Gamma \vdash t : \Pi(x : A).B \qquad \Gamma \vdash u : A}{\Gamma \vdash t @_{x:A.B} u : B[u/x]}$$

~~One application~~ One for each domain A and codomain B (think of semantics).

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Syntax so common that many don't realize that an omission is being made

Typechecking without annotations

Omission has a cost Knowing annotations is needed for typing

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Unification succeeds, with $\alpha_0 = (\alpha_5 \to \alpha_3) \to \alpha_5 \to \alpha_3$

Why does Hindley-Miler type inference works?

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We need a different solution

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Complements unannotated syntax, *locally* explains how to recover annotations

Contribution

Bidirectional type systems have been studied and proposed for many theories

However, general guidelines have remained informal, no unified framework

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1. We give a general definition of type theories (or equivalently, a *logical framework*) supporting non-annotated syntaxes
2. For each theory, we define declarative and bidirectional type systems
3. We show, in a theory-independent fashion, their equivalence

The theories

One syntax for all!

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$$
\begin{aligned}
t, u, T, U ::= & \mid x && \text{(variables)} \\
& \mid \mathsf{x}\{u_1, \ldots, u_k\} && \text{(metavariables)} \\
& \mid c(\vec{x}_1.u_1, \ldots, \vec{x}_k.u_k) && \text{(constructor application)} \\
& \mid d(t;\ \vec{x}_1.u_1, \ldots, \vec{x}_k.u_k) && \text{(destructor application)}
\end{aligned}
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Symbols separated between *constructors* c (intros) and *destructors* d (elims)

In $d(t; \ldots)$, we call t the *principal argument*.

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Example

$$
\begin{aligned}
\Sigma_{\lambda\Pi} = \quad & \Pi(\mathsf{A}, \mathsf{B}\{x\}),\ \lambda(\mathtt{t}\{x\}), \dots && \text{(constructors)} \\
& @(\mathtt{t}; \mathtt{u}) && \text{(destructors)}
\end{aligned}
$$

$$
t, u, A, B ::= x \mid \mathtt{x}\{\vec{t}\} \mid @(t; u) \mid \lambda(x.t) \mid \Pi(A, x.B) \mid \dots
$$

The theories

A *theory* $\mathbb{T}$ is made of *schematic typing rules* and *rewrite rules*.

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Sort rules Sorts are terms that can type other terms[1]

Used to define the *judgment forms* of the theory.

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Example: In MLTT, 2 judgment forms: "$\square$ type" and "$\square : A$" for a type A.

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$$\frac{}{\vdash \mathrm{Ty}\ \mathsf{sort}} \qquad A\ \mathrm{type} \quad \rightsquigarrow \quad A : \mathrm{Ty}$$

$$\frac{\vdash \mathsf{A} : \mathrm{Ty}}{\vdash \mathrm{Tm}(\mathsf{A})\ \mathsf{sort}} \qquad t : A \quad \rightsquigarrow \quad t : \mathrm{Tm}(A)$$

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Constructor rules

Constructors are bidirectionally typed in mode check (its sort is an input)

The sort of the rule should be a *pattern* T^{P}, allowing to recover omitted annotations

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Destructor rules

Destructors are bidirectionally typed in mode infer (the sort is an output)

The sort of the *principal argument* $\mathtt{t} : T^{\mathsf{P}}$ should be a pattern

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Rewrite rules

In type theory, terms should compute

Rewrite rules The computational rules of the theory.

$$@(\lambda(x.\mathtt{t}\{x\});\mathtt{u}) \longmapsto \mathtt{t}\{\mathtt{u}\}$$

In general, of the form $d(t^{\mathsf{P}}; \vec{x}_1.t_1^{\mathsf{P}}, \ldots, \vec{x}_k.t_k^{\mathsf{P}}) \longmapsto r$, with left-hand-side linear.

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Condition: no two left-hand sides unify.

Therefore, rewrite systems are *orthogonal*, hence *confluent* by construction!

Full example, in the *formal* notation

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Theory $\mathbb{T}_{\lambda\Pi}$ A minimalistic type theory with only dependent functions

$\mathrm{Ty}(\cdot)$ sort

$\mathrm{Tm}(\mathsf{A} : \mathrm{Ty})$ sort

$\Pi(\cdot;\ \ \mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty}) : \mathrm{Ty}$

$\lambda(\mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty};\ \ \mathsf{t}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Tm}(\mathsf{B}\{x\})) : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))$

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$\mathrm{Tm}(\mathsf{A} : \mathrm{Ty})$ sort

$\Pi(\cdot;\ \ \mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty}) : \mathrm{Ty}$

$\lambda(\mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty};\ \ \mathsf{t}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Tm}(\mathsf{B}\{x\})) : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}))$

$@(\mathsf{A} : \mathrm{Ty},\ \mathsf{B}\{x : \mathrm{Tm}(\mathsf{A})\} : \mathrm{Ty};\ \ \mathsf{t} : \mathrm{Tm}(\Pi(\mathsf{A}, x.\mathsf{B}\{x\}));\ \ \mathsf{u} : \mathrm{Tm}(\mathsf{A})) : \mathrm{Tm}(\mathsf{B}\{\mathsf{u}\})$

$@(\lambda(x.\mathsf{t}\{x\}); \mathsf{u}) \longmapsto \mathsf{t}\{\mathsf{u}\}$

In the rest of the talk, we use the inference-rule notation for readability ☺

Type systems

Each theory $\mathbb{T}$ defines two type systems.

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We will see each system and show that they are equivalent

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Bidirectional type system

Matching modulo rewriting

Suppose we want to infer the sort of $@(t; u)$

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Given t^{P} and u, we define a matching judgment

$$t^{\mathsf{P}} \prec u \rightsquigarrow \vec{x}_1.t_1/\mathsf{x}_1, \ldots, \vec{x}_k.t_k/\mathsf{x}_k$$

that tries to compute a metavariable substitution s.t. $t^{\mathsf{P}}[\vec{x}_1.t_1/\mathsf{x}_1, \ldots, \vec{x}_k.t_k/\mathsf{x}_k] \equiv u$.

Inferable and checkable terms

Not all unannotated terms can be algorithmically typed

$$\dfrac{\dfrac{?}{\Gamma \vdash \lambda(x.t) \Rightarrow ?} \qquad \cdots}{\Gamma \vdash @(\lambda(x.t); u) \Rightarrow ?}$$

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$$t^{\mathrm{i}}, u^{\mathrm{i}} ::= x \mid d(t^{\mathrm{i}};\ \vec{x}_1.u^{\mathrm{c}}_1, \ldots, \vec{x}_k.u^{\mathrm{c}}_k)$$

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For most theories: $t^{\mathrm{c}}, u^{\mathrm{c}}, \ldots$ = normal forms

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Suppose underlying theory $\mathbb{T}$ is valid.

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Completeness For t^{i} inferable, if $\Gamma \vdash t : T$ then $\Gamma \vdash t^{\mathsf{i}} \Rightarrow U$ with $T \equiv U$.
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Not for one particular theory, but for *all* instances of our framework

More examples

Dependent sums

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty}}{\vdash \Sigma(\mathsf{A}, \mathsf{B}) : \mathrm{Ty}}$$

$$\frac{\begin{array}{cc}\vdash \mathsf{A} : \mathrm{Ty} & x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \\ \vdash \mathsf{t} : \mathrm{Tm}(\mathsf{A}) & \vdash \mathsf{u} : \mathrm{Tm}(\mathsf{B}\{\mathsf{t}\})\end{array}}{\vdash \mathrm{pair}(\mathsf{t}, \mathsf{u}) : \mathrm{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \\ \vdash \mathsf{t} : \mathrm{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))\end{array}}{\vdash \mathrm{proj}_1(\mathsf{t}; \cdot) : \mathrm{Tm}(\mathsf{A})}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \\ \vdash \mathsf{t} : \mathrm{Tm}(\Sigma(\mathsf{A}, x.\mathsf{B}\{x\}))\end{array}}{\vdash \mathrm{proj}_2(\mathsf{t}; \cdot) : \mathrm{Tm}(\mathsf{B}\{\mathrm{proj}_1(\mathsf{t})\})}$$

$$\mathrm{proj}_1(\mathrm{pair}(\mathsf{t}, \mathsf{u}); \varepsilon) \longmapsto \mathsf{t} \qquad \mathrm{proj}_2(\mathrm{pair}(\mathsf{t}, \mathsf{u}); \varepsilon) \longmapsto \mathsf{u}$$

Lists

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathtt{A} : \mathbf{Ty}}{\vdash \mathbf{List}(\mathtt{A}) : \mathbf{Ty}} \qquad \frac{\vdash \mathtt{A} : \mathbf{Ty}}{\vdash \mathbf{nil} : \mathbf{Tm}(\mathbf{List}(\mathtt{A}))} \qquad \frac{\vdash \mathtt{A} : \mathbf{Ty} \qquad \vdash \mathtt{x} : \mathbf{Tm}(\mathtt{A}) \qquad \vdash \mathtt{l} : \mathbf{Tm}(\mathbf{List}(\mathtt{A}))}{\vdash \mathbf{cons}(\mathtt{x}, \mathtt{l}) : \mathbf{Tm}(\mathbf{List}(\mathtt{A}))}$$

$$\frac{\vdash \mathtt{A} : \mathbf{Ty} \qquad \vdash \mathtt{l} : \mathbf{Tm}(\mathbf{List}(\mathtt{A})) \qquad x : \mathbf{Tm}(\mathbf{List}(\mathtt{A})) \vdash \mathtt{P} : \mathbf{Ty} \qquad \vdash \mathtt{pnil} : \mathbf{Tm}(\mathtt{P}\{\mathbf{nil}\}) \qquad x : \mathbf{Tm}(\mathtt{A}), y : \mathbf{Tm}(\mathbf{List}(\mathtt{A})), z : \mathbf{Tm}(\mathtt{P}\{y\}) \vdash \mathtt{pcons} : \mathbf{Tm}(\mathtt{P}\{\mathbf{cons}(x, y)\})}{\vdash \mathbf{ListRec}(\mathtt{l}; \mathtt{P}, \mathtt{pnil}, \mathtt{pcons}) : \mathbf{Tm}(\mathtt{P}\{\mathtt{l}\})}$$

$$\mathbf{ListRec}(\mathbf{nil}; x.\mathtt{P}\{x\}, \mathtt{pnil}, xyz.\mathtt{pcons}\{x, y, z\}) \longmapsto \mathtt{pnil}$$

$$\mathbf{ListRec}(\mathbf{cons}(\mathtt{x}, \mathtt{l}); x.\mathtt{P}\{x\}, \mathtt{pnil}, xyz.\mathtt{pcons}\{x, y, z\}) \longmapsto$$
$$\mathtt{pcons}\{\mathtt{x}, \mathtt{l}, \mathbf{ListRec}(\mathtt{l}; x.\mathtt{P}\{x\}, \mathtt{pnil}, xyz.\mathtt{pcons}\{x, y, z\})\}$$

W types

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty}}{\vdash \mathrm{W}(\mathsf{A}, \mathsf{B}) : \mathrm{Ty}} \qquad \frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \\ \vdash \mathsf{a} : \mathrm{Tm}(\mathsf{A}) \qquad \vdash \mathsf{f} : \mathrm{Tm}(\Pi(\mathsf{B}\{\mathsf{a}\}, x'.\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})))\end{array}}{\vdash \mathrm{sup}(\mathsf{a}, \mathsf{f}) : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\}))}$$

$$\frac{\begin{array}{c}\vdash \mathsf{A} : \mathrm{Ty} \qquad x : \mathrm{Tm}(\mathsf{A}) \vdash \mathsf{B} : \mathrm{Ty} \qquad \vdash \mathsf{t} : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})) \qquad x : \mathrm{Tm}(\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\})) \vdash \mathsf{P} : \mathrm{Ty} \\ x : \mathrm{Tm}(\mathsf{A}), y : \mathrm{Tm}(\Pi(\mathsf{B}\{x\}, x'.\mathrm{W}(\mathsf{A}, x.\mathsf{B}\{x\}))), z : \mathrm{Tm}(\Pi(\mathsf{B}\{x\}, x'.\mathsf{P}\{@(y, x')\})) \vdash \mathsf{p} : \mathrm{Tm}(\mathsf{P}\{\mathrm{sup}(x, y)\})\end{array}}{\vdash \mathrm{WRec}(\mathsf{t}; \mathsf{P}, \mathsf{p}) : \mathrm{Tm}(\mathsf{P}\{\mathsf{t}\})}$$

$$\mathrm{WRec}(\mathrm{sup}(\mathsf{a}, \mathsf{f}); x.\mathsf{P}\{x\}, xyz.\mathsf{p}\{x, y, z\}) \longmapsto \mathsf{p}\{\mathsf{a}, \mathsf{f}, \lambda(x.\mathrm{WRec}(@(\mathsf{f}, x); x.\mathsf{P}\{x\}, xyz.\mathsf{p}\{x, y, z\}))\}$$

Universes

Extends $\mathbb{T}_{\lambda\Pi}$ with

$$\frac{}{\vdash \mathrm{U}(\cdot) : \mathrm{Ty}} \qquad \frac{\vdash \mathsf{a} : \mathrm{Tm}(\mathrm{U})}{\vdash \mathrm{El}(\mathsf{a};\cdot) : \mathrm{Ty}}$$

Tarski-style Adds codes for all types

$$\frac{}{\vdash \mathrm{u}(\cdot) : \mathrm{Tm}(\mathrm{U})} \qquad \mathrm{El}(\mathrm{u};\varepsilon) \longmapsto \mathrm{U}$$

$$\frac{\vdash \mathsf{a} : \mathrm{Tm}(\mathrm{U}) \qquad x : \mathrm{Tm}(\mathrm{El}(\mathsf{a})) \vdash \mathsf{b} : \mathrm{Tm}(\mathrm{U})}{\vdash \pi(\mathsf{a},\mathsf{b}) : \mathrm{Tm}(\mathrm{U})}$$

$$\mathrm{El}(\pi(\mathsf{a}, x.\mathsf{b}\{x\});\varepsilon) \longmapsto \Pi(\mathrm{El}(\mathsf{a};\varepsilon), x.\mathrm{El}(\mathsf{b}\{x\};\varepsilon))$$

(Weak) Coquand-style

Adds a code constructor c

$$\frac{\vdash \mathsf{A} : \mathrm{Ty}}{\vdash \mathrm{c}(\mathsf{A}) : \mathrm{Tm}(\mathrm{U})}$$

$$\mathrm{El}(\mathrm{c}(\mathsf{A});\varepsilon) \longmapsto \mathsf{A}$$

Conclusion

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Generic account of bidirectional typing for class of dependent type theories

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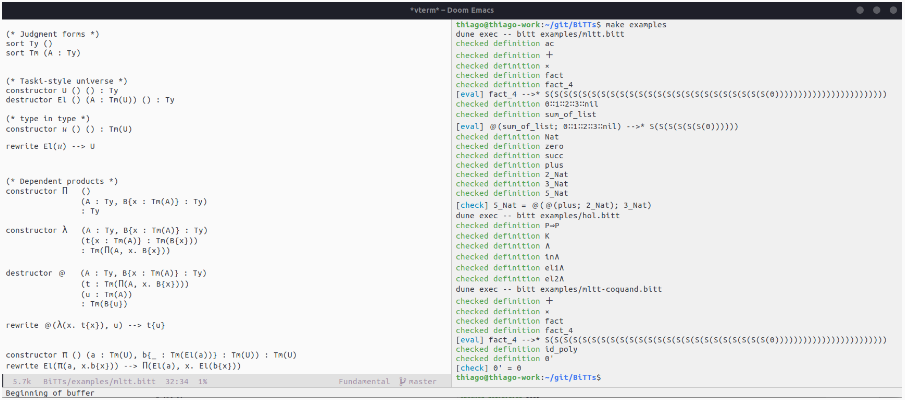
Bidirectional system implemented in a prototype, available at

`https://github.com/thiagofelicissimo/BiTTs`



Thank you for your attention!